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# Analyzing Second-Year University Students' Rational Number Understanding: A Case on Interpreting and Representing Fraction

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**Abstract:** This research aims to determine second-year university students' understanding in interpreting and representing fractions. A set of fraction tests was given to students through two direct learning interventions. An unstructured interview was used as an instrument to obtain explanations and confirmations from the purposive participants. A total of 112 student teachers of primary teacher education program at two private universities in Indonesia were involved in this research. A qualitative method with a holistic type case study design was used in this research. The results indicate that a significant percentage of the participants could not correctly interpret and represent fractions. In terms of interpretation, it is found how language could obscure the misunderstanding of fractions. Then, the idea of a fraction as part of a whole is the most widely used in giving meaning to a fraction compared to the other four interpretations, but with limited understanding. Regarding data representation, many participants failed to provide a meaningful illustration showing the improper fraction and mix number compared to the proper fraction. Improvement of fraction teaching at universities - particularly in primary teacher education programs - is needed so that students get the opportunity to develop and improve their knowledge profoundly. We discuss implications for teaching fractions.

**Keywords:** *Interpreting fraction, rational number, representing fraction.*

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## Introduction

Fractions as objects of calculation (Kieren, 1976) and phenomenological forms of rational numbers (Freudenthal, 2002) are one of mathematics topics taught at the elementary school level (Indonesia Ministry of National Education and Culture, 2016; National Council of Teachers of Mathematics [NCTM], 2015). Understanding the concept of fractions is crucial for students in building the foundation for their numerical, arithmetic, algebraic, and proportional reasoning development (Lazić et al., 2017; Obersteiner et al., 2019; Siegler & Braithwaite, 2017; Siegler & Forgues, 2017; Siegler & Pyke, 2013). Teachers must have a broader and more substantial knowledge of what is taught (content knowledge of fraction) and how it is taught (pedagogical content knowledge of fraction) to their students (Pournara et al., 2015; Santagata & Lee, 2021; Tian & Siegler, 2018). However, it is undeniable that fractions are still one of the the central topics in mathematics which are difficult for students (Forgues et al., 2015; Lestari et al., 2020; Mastuti, 2017; Obersteiner et al., 2019; Wijaya, 2017; Yetim & Alkan, 2013), and many teachers find it difficult to understand and teach (Klemer et al., 2019; Lee et al., 2011).

University students of primary teacher education program (trained as general classroom teachers) who are studying at undergraduate level of the elementary school teacher program must master the basic concepts of mathematics (in elementary school) well and fluently (NCTM, 2012, 2015). This demand, of course, has a powerful reason, considering the role of teacher professionalism that they will play in the future in teaching mathematics concepts to students (Damrongpanit, 2019; Depaepe et al., 2015; Webster, 2020). Related to fractions, some previous studies have highlighted final-year university students' or preservice teachers' knowledge of the concepts and procedures on rational numbers (Osana & Royea, 2011; Vula & Kingji-Kastrati, 2016). The trend of research focuses on revealing how mathematical content knowledge affects preservice pedagogy (Castro-Rodríguez & Rico, 2021; Depaepe et al., 2015) and noticing skill in responding to reactions/errors made by students (Ivars et al., 2018), or how mathematical concept

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knowledge affects preservice teachers' performance in doing number operations (Leung & Carbone, 2013; Putra, 2016; Webster, 2020). A common important assumption generated by them all is that there were still barriers to acquiring rational numbers, especially fractional content and concepts, caused by a lack of understanding. This condition is closely related to the preservice teacher's previous learning experience. Most of the recent research has focused on uncovering the obstacles faced by students at school levels (Forgues et al., 2015; Lemonidis & Pilianidis, 2020; Wahyu et al., 2020). However, only a few studies explored the difficulties experienced by first and second-year students of the primary school teacher education program in understanding fractions. Such research will provide an insight into the actual challenges that university students encounter related to fractions.

The fact that fractions can have many meanings is also a significant source of difficulties for students learning fractional concepts (Musser et al., 2011; Pitta-Pantazi, 2014). Therefore, knowledge about interpreting and representing fractions is an initial basic concept that students must understand to advance in further fractions such as equivalent, equality, density, and performing operations (Chapin & Johnson, 2006). Students who only know that fractions are part-to-whole relationships will undoubtedly have a poor understanding of the whole fraction concept (Lamon, 2020; Lazić et al., 2017).

Knowing students' knowledge in interpreting and representing fractions will give an idea of their understanding of fractions (Kang & Liu, 2018; Leung & Carbone, 2013) and provide us with a sense of how they can validate their knowledge when teaching fractions (Lee et al., 2011; Webster, 2020). The description of students' understanding about mathematical concepts, particularly fractions, will be used to assess the extent to which the learning process designed in the primary school teacher education program has impacted the development of students' mathematical knowledge (Viseu et al., 2020). In addition, this will also be taken into consideration in deciding what steps to take to improve and develop mathematics learning programs in the primary school teacher education program.

This study aims to present the second-year student teachers' understanding of interpreting and representing fractions correctly. The types of interpretation used to describe the meaning of fractions and the reason given by students were identified and explained. Likewise, the way they represent various symbols in some visual models was also examined. We did not give directions regarding the form of illustration they used. In simple terms, this research focuses on how students can interpret and represent fractions meaningfully. Therefore, the following research question guided the study: How is the second-year student teachers' understanding in interpreting and representing fractions?

The study extends previous research by providing insights into university students' perceptions of what they face in their struggle to interpret fractions. This contributes to better mathematics teaching in teacher education programs.

## Literature Review

*Interpretation and Representation of Fraction*

Fraction is more commonly used in showing or representing rational numbers in the school curriculum (Musser et al., 2011; Siegler & Forgues, 2017). Developing an understanding of fractions is complex because fractions have multiple interpretations (Forgues et al., 2015; Kieren, 1976). Kieren (1976) originally introduced the idea of seven sub-constructs of the rational number and later revised these to four sub-constructs based on part-whole conceptions, which included ratios, quotients, measures, and multiplicative operators (Kieren, 1980). The four fractional constructs are interrelated, and each construct allows the consideration of rational numbers from a different perspective (Behr et al., 1983; Pitta-Pantazi, 2014).

Kieren's idea of part-whole and ratio are related. In both interpretations, fractions are interpreted as quantifying the relationship between the whole and a specified number of parts. These relationships are phenomenally expressed in set-subset, dissected and shaded regions, and number line relationships. It means that the interpretation of fractions as part-whole relationships, for both continuous and discrete objects, states the relationship between parts and all parts of the same size (unit partitioned into equal-size parts) and the relationship set partitioned into equal-size groups (Kennedy et al., 2011). When rational numbers x (which satisfies $bx = a$, or $x=\frac{a}{b}$ or a/b, or ${}^{a}/_{b}$ where $a, b \in Z, b \neq 0$) is introduced as fractions that represent a part of a whole, we must pay attention to the whole from which a rational number or fraction is derived. We should at least consider these three things:(1) the whole being considered; (2) the number b of equal-size parts into which the whole has been divided; (3) the number *a* of parts of the whole that are selected (Billstein et al., 2014; Kieren, 1980). One thing to remember is that ratios do not always follow the same rules as fractions if we talk about ratios themselves. A ratio is a comparison between two quantities. When a ratio compares a part to a whole, the part-to-whole interpretation of a fraction is being used (Chapin & Johnson, 2006).

The third idea, namely fractions as quotients, is also closely related to the part-whole relationship. Quotient interpretation considers a fraction as the result of dividing an object or a specific integer by an integer other than zero, e.g., 2:3 = 2/3. For this interpretation, students should be able to identify fractions with division and understand the role of the dividend and the divisor in this operation (Chapin & Johnson, 2006; Musser et al., 2011).

*Measure interpretation* is usually carried out through an iteration of counting the number of whole units usable in "covering" the region. In addition, measure interpretation is often associated with the position of a fraction on a number line to represent the size or value of a unit fraction (a unit fraction is identified, e.g., 1/3), such as route distance, and so on (Freudenthal, 2002; Lamon, 2020; Musser et al., 2011). In this interpretation, the students should use a given unit interval to measure any distance from the origin (e.g., 2 x 1/3 = 2/3), locate a number on a number line, and identify a number represented by a point on the number line. Further *to the operator idea*, this sub-construct focuses on fractions as elements in the algebra of functions, e.g., showing 2/3 of a pie chart or finding 2/3 of 12. The composition of operators provides an elementary foundation for the multiplication of rational numbers. Table 1 presents a brief description of five ideas to interpret fractions.

*Table 1. Example of Various Interpretations of the Fraction 2/3*

| No | Interpretation | | e.g., descriptions |
|---|---|---|---|
| 1 | part-whole relationship | unit partitioned into equal-size parts | 2 out of 3 equal parts of a whole of a rectangle |
| | | set partitioned into equal-size Groups | 2 out of 3 equal groups/collections of wholes of set |
| 2 | Quotient | | Two divided by 3, so $\frac{2}{3}$ is the amount each person receives |
| 3 | Ratio | | Two parts out of every 3 are green |
| 4 | Measure | | $\frac{2}{3}$ means a distance of 2 ($\frac{1}{3}$units) from 0 on the number line |
| 5 | Operator | | $\frac{2}{3}$ of something, stretching or shrinking |

Mathematical representation is defined as visible or tangible productions that encode, stand for, or embody mathematical ideas or relationships (Goldin, 2014). The term representation is also used to refer to a person's mental or cognitive constructs, concepts, or configurations. In teaching and learning fractions, the use of manipulatives (concrete or virtual) and numerous representations is regarded as a critical aspect (Goldin, 2014; Kang & Liu, 2018).

Representation can help students comprehend mathematics (Brijlall et al., 2012) and make it easier to analyze problems and develop solution strategies (Westenskow et al., 2014). Therefore, teachers or student teachers must clearly understand how multiple representations can be used to assist student learning (Lemonidis & Pilianidis, 2020) and effectively set the basis for fractional understanding and learning (Damrongpanit, 2019; Webster, 2020). This responsibility is closely related to teachers' central role as creators of the effective and quality mathematics learning environment (NCTM, 2014; Santagata & Lee, 2021).

### *Teachers' and Student Teachers' Knowledge of Fractions*

As prospective teachers, student teachers in the primary school teacher education program must thoroughly comprehend the content knowledge to be taught, including mathematics (NCTM, 2014, 2015). Much of the recent research in mathematics education has been focused on pre-service and in-service teacher knowledge in mathematics (Depaepe et al., 2015; Ivars et al., 2018, 2020; Vula & Kingji-Kastrati, 2016). This trend is linked to efforts to improve students' achievement and understanding of mathematics. Within these standards, teachers' mathematical knowledge is one of the critical components of teaching effectiveness and plays a crucial role in student achievement (Pournara et al., 2015; Santagata & Lee, 2021).

However, studies also address the prospective teachers' limited grasp of mathematics content knowledge and their teaching competence (Castro-Rodríguez & Rico, 2021; Depaepe et al., 2015; Klemer et al., 2019). Related to rational numbers, some researchers discovered that pre-service and in-service teachers lacked sufficient mathematics content knowledge and struggled to teach it (Lazić et al., 2017; Ni & Zhou, 2005; Osana & Royea, 2011); limited knowledge of proper procedures to solve rational number problems and were not able to describe reasons for their answers (Leung & Carbone, 2013; Putra, 2016); or preferred to solve rational number problems using the procedural approach over the conceptual approach (Vula & Kingji-Kastrati, 2016). All of these issues were caused by one or more factors. For example, the learning experience may not meet genuine pedagogical needs (Webster, 2020).

For this reason, investigating the learning difficulties faced by student teachers in first- or second-year universities is crucial. Since, the difficulties do not appear right away when learning complex rational number arithmetic. Instead, they arise when students learn about fundamental ideas of number symbols, particularly the meaning of fractions.

## Methodology

### *Research Design*

This study uses a qualitative method with a holistic type of case study design (single unit of analysis) (Yin, 2018). Case study research begins with identifying certain cases to be described and analyzed (Creswell & Poth, 2017). In this study, the cases or conditions discussed are only focused on the knowledge of students in interpreting and representing fractions. The qualitative method used directs the researcher to produce various written and oral descriptions of the participants' behavior related to interpreting and representing fractions observable in a social situation designed by the researcher (Fraenkel et al., 2012; Taylor & Bogdan, 2015).

The qualitative research procedure carried out in this study followed six steps as presented by Fraenkel et al. (2012). *The first step* was identifying the problem to be studied - how is the second-year student teachers' understanding in interpreting and representing fractions? The researcher formulated the problem clearly so that the research was directed to search for the solution. *In the second step*, identifying the participants. Then, *the third step* is formulating research hypotheses. The researchers formulated initial assumptions related to this research where students still have problems in understanding fractions. *In the fourth step*, collecting data, the researchers collected data using a set of written tests and unstructured interviews. *The fifth step*, data analysis, analyzing the data involved a coherent description of what was observed and discovered (Ary et al., 2014). *In the sixth step*, interpretation and conclusion, the researcher continuously carried out this last stage until the researcher identified students' knowledge in interpreting and representing fractions.

*Sample and Data Collection*

This study involved 112 undergraduate students of primary teacher education program at two private universities in Indonesia. Of the total participants, 97 (86.6%) were female, and 15(13.4%) were male and their ages ranged between 19- and 21-years old. All of the participants have the same characteristics: they have taken 4-5 credits of mathematics courses provided in the program curriculum.

The data were collected by giving the set of tests in two interventions using direct learning. Unstructured interviews were also used as the instrument (Creswell, 2012) to gain primary data about how students applied their mathematics knowledge about fractions. The test instrument consisting of eight questions was adapted from related research about fractions (Leung & Carbone, 2013). Table 2 shows the tasks.

*Table 2. Shows the Test Given to the Participants.*

| No | Problems/Code | General Goal |
|---|---|---|
| 1. | Task 1: write the meanings of the following fraction symbols! (Pose as many different situations or illustration to support your answer)<br>a. $\frac{1}{2}$ (M1) c. $1\frac{1}{3}$ (M3)<br>b. $\frac{3}{4}$ (M2) d. $\frac{4}{3}$ (M4) | Exploring the students' knowledge about the meaning of fractions from various symbols/types of fractions (proper, improper, and mixed numbers) |
| 2. | Task 2: sketches as many different (not being similar to each other) pictures as you can using the following fractions!<br>a. $\frac{1}{2}$ (D1) c. $6\frac{3}{4}$ (D3)<br>b. $\frac{2}{3}$ (D2) d. $\frac{4}{3}$ (D4) | Exploring the students' knowledge about illustrated the images or conditions represented from the given fraction symbol, e.g., shading parts in geometrical shapes, taking equal-size groups of the set objects, or creating the number line. |

In the interview process (which lasted between 10 to 15 minutes), the researchers offered some questions related to student responses to confirm their understanding and thinking about fractions, e.g., *what does your statement mean? What prompted you to respond in this manner? Do you believe the picture you drew represents this symbol (fractions)?* Each interview was audio-recorded, and a verbatim transcript was provided.

*Analyzing of Data*

Quantitative data, the number of participants who completed a series of tests, were analyzed using descriptive statistics. Then qualitative data - in the form of students' strategies in completing the test and their thinking processes which were told in the interviews - were used to follow up on the quantitative results and provide an overview of students' knowledge in interpreting and representing fractions.

Qualitative analysis was carried out using three stages of interactive data analysis techniques (Miles et al., 2014), namely (1) *data condensation* – which refers to the process of selecting (coding), focusing, simplifying, abstracting, and/or transforming the data that appear in the participants' worksheet or answer sheet, written-up field notes, interview transcripts, documents, and other empirical materials. During the coding process, Nvivo 12 Plus software was utilized as a tool to help categorize data according to themes systematically; (2) *data display*- at this stage, the researcher showed information that had been well organized and compressed, allowing drawing conclusions and actions. In this study, the students' answers written on the answer sheet or worksheet were described and displayed. Assessment of the success of both interpretations and representations was done by looking at the fulfillment of at least one correct or appropriate interpretation or representation, and (3) *conclusion drawing & verifying* - the conclusion was drawn based on the data obtained. The conclusion of this study was the answer to the research question posed - a description of the students' knowledge in interpreting and representing fractions.

In order to increase the trustworthiness of the data, triangulation and member checks techniques were used, as suggested by Ary et al. (2014). To examine the inter-rater reliability, two researchers independently conducted the content analysis to code the data and determine that all the coding was appropriate and fit into the proper category. The rate of agreement on the coding of the responses (*kind of errors in interpreting fractions, kinds of fraction interpretations, features of model representations*) was between 94 and 97%. The responses to which the disagreement occurred were reread, and an agreement was reached.

## Results

The results are presented based on the data gathered during the research, which include the participants' answers and script interviews with selected participants. In general, Table 3 shows the results of the quantitative analysis of the participants' knowledge in interpreting and representing fractions. On average, 32.14% of students succeeded in interpreting the fractions and 47.32% succeeded in representing the fractions. Although the percentage of representation successful was significant compared to interpreting fractions, both are still less than half of the total participants. Specifically, less than a quarter of the participants was successful both in interpreting and representing the improper and mixed fraction.

*Table 3. Frequencies and Success Rates, for the First Answer of Students for Each Problem (Source Primary Data)*

| No | Problems related to | | Percentage of Interpretation | | Percentage of Representation | |
|---|---|---|---|---|---|---|
| | | | **Problem** | **success** | **Problem** | **success** |
| 1. | Proper Fraction | Single/Unit Fraction | M1 | 28,57 | D1 | 85.71 |
| | | Non-Unit Fraction | M2 | 56,25 | D2 | 72.32 |
| 2. | Mixed Fraction | | M3 | 23,21 | D3 | 13.39 |
| 3. | Improper Fraction | | M4 | 20,54 | D4 | 18,75 |
| | | | N=112 Mean | 32.14 | N=112 Mean | 47.54 |

The low percentage of the participants' success in interpreting and representing fractions means that more than 50% still have difficulties with both. This condition is essential and exciting to discuss to get an idea of the issues experienced by students in struggling to learn fractions. Table 4 provides an overall description of the errors made by all participants in interpreting the M1-M4. The coding process (using Nvivo 12) that was carried out openly and systematically on the mistakes made by the participants resulted in 6 themes:

*Table 4. Kind of Errors in Interpreting Fraction Based on Data Analysis*

| No | Kinds of Error in Interpreting Fraction & Their Description | | Percentage of Error M1 | M2 | M3 | M4 |
|---|---|---|---|---|---|---|
| 1. | Ɛ1 | *Ɛ1* for *the wrong explanation*, such as:<br>▪ misinterpreting the part of the whole *(e.g., ignoring the concept of fractions as "equal parts," not understanding the meaning of the denominator as the name of the objects (part or subset) and the numerator as the number of the objects (or set) or number the partition of the whole*);<br>▪ misinterpreting the fraction as a ratio;<br>▪ using the wrong term (*e.g., ½ as one-half of one or ¾ as three taken from four*)<br>▪ misrepresents improper fraction (*e.g., 4/3 as 4 taken from three*) | 16.07 | 6.25 | 38.39 | 35.71 |
| 2. | Ɛ2 | *Ɛ2* for recall, writing verbal words for naming the fraction (*e.g., one by two, one per two, three per three, etc. without any explanation*) | 31.25 | 11.61 | 14.29 | 16.07 |
| 3. | Ɛ3 | *Ɛ3* for writing a fraction as numerator and denominator | 13.39 | 12.50 | 0 | 6.25 |
| 4. | Ɛ4 | *Ɛ4* to interpret fraction by<br>▪ writing the kind of fraction (*e.g., it was a proper fraction or mixed number*)<br>▪ writing the definition of a rational number (*e.g., it was a form of a/b, where a and b were natural numbers and b≠0*) | 5.36 | 4.46 | 9.82 | 9.82 |
| 5. | Ɛ5 | *Ɛ5* for other representation (*e.g., writing representation in decimal or percentage form without explanation*) | 3.57 | 6.25 | 0 | 0.89 |
| 6. | Ɛ6 | *Ɛ6* for not answering at all | 1.79 | 2.68 | 14.29 | 10.71 |

Toward the types of errors $\mathcal{E}2$ to $\mathcal{E}6$, it is clear what the participants were doing, such as interpreting a fraction by simply using verbal words (*e.g., One-half, one-one third, three-quarter*) or writing the numerator and denominator (*e.g., the numerator is 3 and the denominator is 4).* Of course, this kind of error should no longer happen to students considering that they have studied fractions at the previous level. Even in the case of this study, all participants had attended mathematics courses in their first and second years. So, the development of the mathematics learning process that focuses on mathematical concepts in both university and school must be emphasized. In the following discussion section, all of the emerging participants' strategies in interpreting and representing fractions as well as their mistakes are explored. We believe that this description gives us an idea of how the participants perceive fractions and struggle to understand fractions.

## Discussion

*Students' Knowledge on Interpreting Fractions*

Interpreting fractions is an initial concept in learning fractions that students must understand well for fluency in order to grasp additional concepts such as equivalent, density, and operations on fractions (Chapin & Johnson, 2006). Following the Indonesian curriculum, students study fractions from elementary to middle school level. In elementary school, various concepts of fractions are introduced, such as fractions as part of a whole, presenting fractions on a number line, to operations on fractions. In high school, operations and properties of fractional operations involving algebraic topics are given to students. As a result, students at the higher education level are required to have advanced understanding and comprehension of fractions.

However, the findings of this study indicate that the participants' knowledge, particularly in interpreting fractions, is still quite limited. The perception of fractions is the solely part-whole relationship (equal-size parts), while the other four interpretations are rarely or never indicated by the participants.

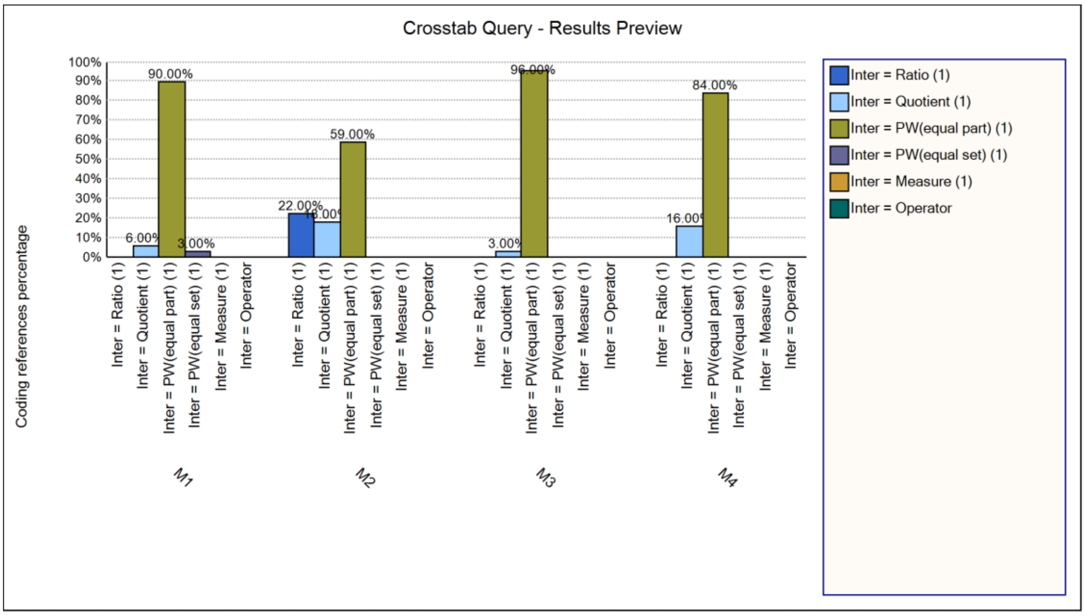
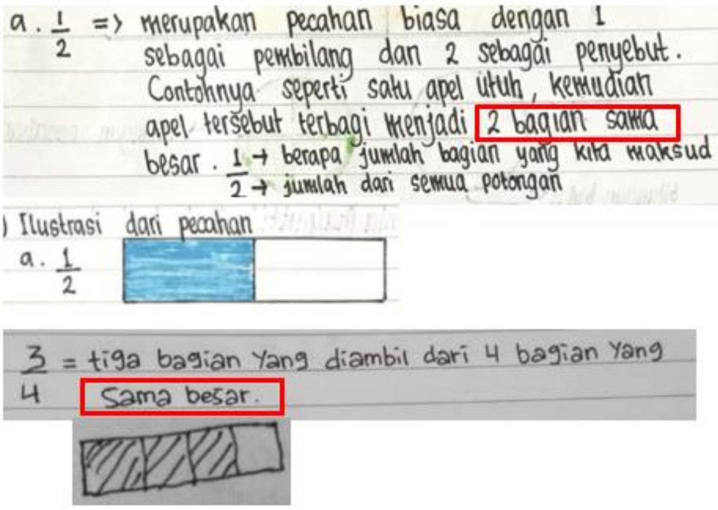



*Figure 1. Kinds of Fraction Interpretations of M1-M3*

*(Source: Crosstab Query Analysis with Nvivo 12 Plus 2020)*

Figure 1 shows that interpretation of fractions as a part-whole relationship (equal-size parts) was the most frequently used (90% for M1, 59% for M2, 96% for M3, and 84% for M4) in interpreting fractions compared to the parts-whole relationship as equal-size groups (3% for M1) or other interpretation. This finding describes that the participants are more familiar with using parts-whole interpretation than other interpretations. This condition, of course, was heavily impacted by the participants' prior fraction learning experience. We can see the participants' answers from these two interpretations in Figure 2a and Figure 2b.

Knowledge of "*parts have to be the same size (equivalent in size/congruent parts/identical parts)*" is a fundamental concept in studying fractions as part of the whole (Musser et al., 2011). The term equal-size parts or equivalent parts means equivalent in some attributes, such as length, area, volume, number, or weight, depending on the whole and appropriate parts (Musser et al., 2011). However, this knowledge is sometimes neglected or even forgotten when studying fractions.



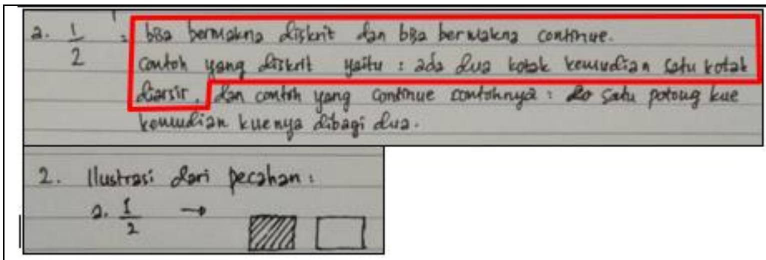

*"It is a proper fraction with 1 as the numerator and 2 as the denominator. For example: like a whole apple, then the apple is divided into 2 equal parts."*

(Participant #40)

*"Three parts taken from 4 equal parts"*

(Participant #58)

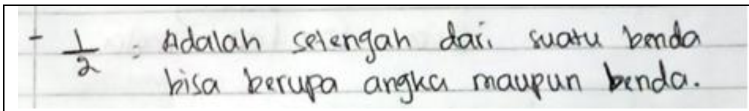
*Figure 2a. Sample Correct Answers in Interpreting and Representing Fractions as Part-Whole Relationship (as equal-size parts)*



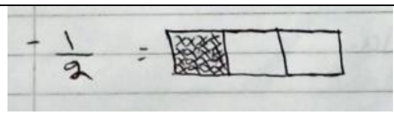

*"=...can mean discrete and can mean continue. A discrete example is: there are two boxes then one box is shaded.*

(Participant #42)

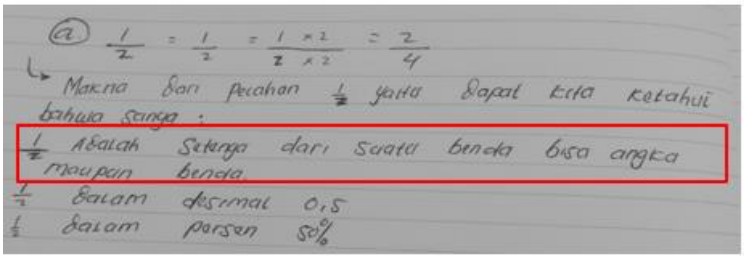
*Figure 2b. Sample Correct Answer in Interpreting and Representing Fraction as Part-Whole Relationship (as an equal-size group)*



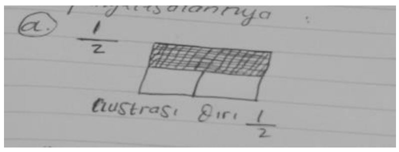

*½: is half of an object can be a number or object*



Wrong illustration of ½
(Participant #34)

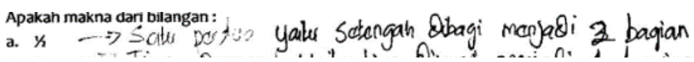



*½: is half of an object can be a number or object*

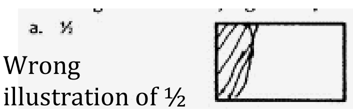


Wrong illustration of ½

(Participant #63)




*½: one per two is half was divided into 2 parts*



Wrong illustration of ½
(Participant #103)

*Figure 3. Sample Incorrect Answers in Interpreting and Representing the M1 Showing Students' Lack of Understanding of the Fraction Concept*

The participants' judgments of fractions that are less responsive to the concept of similarity from each part include (some examples of responses):

> *"1/2 means half of an object, such as a banana which is divided/cut into two parts."* (Participant#30)
>
> *"1/2 means one of two parts."* (Participant#97)
>
> *"3/4 is an ordinary (proper) fraction, for example, a pizza that is cut into 4 parts and you want to use only 3 parts, so the fraction is"* (Participant#31)
>
> *"3/4 means three parts out of a total of 4 parts."* (Participant#50)

The lack of focus on the concept of the denominator as equivalent parts or components eventually leads to incorrect meaning and ambiguity in the meaning grasped by the students. As a result, we found a variety of incorrect explanations and illustrations (see Figure 3). When 1/2 is regarded as one divided by two or an object divided by 2, participants are very likely to be perplexed. Participant #103's illustration in Figure 3 shows that knowledge of the concepts of part of whole and quotient is still distorted. This finding refutes what was conveyed by Wahyu et al. (2020). In his research, students' understanding of fair-sharing does not help them in understanding the unit rate.

Other errors made by participants is due to a lack of understanding of the concept of fractions enclosed: participants tend to use the term "*half of an object*" rather than calling it 1 out of 2 equivalent parts. Although this perception of the meaning appears correct, something unexpected occurred when they were asked to represent based on their understanding (see Figure 3). The participants did not understand the meaning of "half." The word half was used by them because it was used in everyday life (obtained from interviews with participants). This situation highlights how language plays a significant role in understanding fractions (Siegler & Forgues, 2017).

To see how students think when illustrating the number, the interviews were conducted by researcher (R) with Participant#34 (P34) and Participant#63 (P63) delivered through the following script (related to their answer in Figure 3).

*<u>Interview with Participant#34:</u>*

*R : "Do you believe the picture you drew was a representation of ½?"*

*P34 : "Yes,......!"*

*R : "please, try to explain it to me."*

*P34 : "right, half is half thing... so this one (pointing to the shaded one) is one, while this one (pointing to the unshaded one) is two. So this is half."."*

*R : "So which half is it?"*

*P34 :"..emmm...this is one (pointing to the (all) illustration she drew"*

*R :...... (continuing the discussion to give a correct understanding of the meaning of ½)*

*<u>Interview with Participant#63</u>*

*R : "Do you believe the picture you drew was a representation of ½?"*

*P63 : "emmm.... I think so....! Is it wrong, ma'am?"*

*R : " eemmm...let's see first... explain to me, where is the fraction ½?"*

*P63 : "this is ma'am... (points to the picture she made)."*

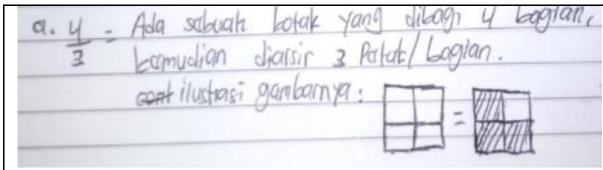
*R :" all of this??"*

*P63 : "Yes. so half is like this ... there is one cake then divided in two to make half ..."*

*R : ... (continuing the discussion to give a correct understanding of the meaning of ½)*

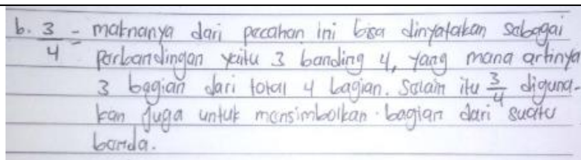
The discussion with Participant#34 and Participant#63 revealed that the understanding of fractions from these two participants was still weak. The notation *a/b* is only seen as a symbol from two numbers (*a* and *b*) separated by a line rather than as a single number. According to Billstein et al. (2014), this perception is very likely to happen when fractions introduce rational numbers as a numeral in the form *a/b*, *a* and *b* are whole numbers and *b≠0*, without further explanation (regarding the relationship that *a* and *b* have as *the number of parts of the whole that are selected* and *the number of equal-size parts into which the whole has been divided).* This bias sometimes occurs because fraction and rational numbers are associated with their whole number knowledge. Then, it becomes a manifestation of confusion between fraction and integer symbols (Ni & Zhou, 2005).

Furthermore, the understanding of fractions, which is only restricted to part-whole relationships, still leaves a gap for difficulties in grasping the interpretation and representation of the improper fraction (M4) symbol. We found that participants who succeeded in interpreting and representing proper fractions (using part-whole relationships) were not necessarily able to interpret improper fractions. This finding confirms what was stated by Kerslake (as cited in Lenz & Wittmann, 2021, p. 2) that students with a good understanding of the interpretation of part-whole fractions may still have a limited view of fractions as numbers and have cognitive difficulties. There are various interpretations and illustrations that the participants gave to show 4/3. Misinterpretations of improper fraction 4/3 include "*four parts of three*," or "*an object which is divided into 4 parts and then shaded by 3 parts*", or "*4 compare to 3*". The students struggled with the meaning of the fractions by giving an inappropriate explanation. The same case is also found in the illustration of the improper number; some even say that "*4/3 cannot be illustrated because the numerator has a value greater than the denominator*".




*"There is a box that is divided into 4 parts, then shaded by 3 plots/sections."*

*(4a)*




*"The meaning of this fraction is as a comparison, namely three to 4 which means 3 parts out of a total of 4 parts, besides that is also used to symbolize part of an object"*

*(4b)*

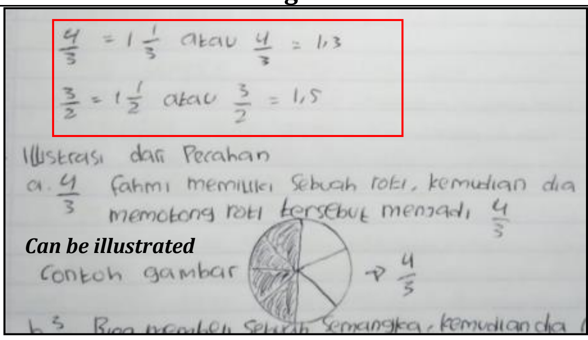
*Figure 4. Participants Used the Meaning of Numerator and Denominator Incorrectly.*

*(Source: primary data, contribution of participant#6)*

One of the identifications that the participants do not understand the concept is not knowing what it is not and when it does not apply (Lamon, 2020). This indication can be seen from the answers given by Participant#6 in Figure 4. He didn't realize that his explanation and representation for 4/3 (see figure 4a) would contradict with ¾ (see figure 4b). Participant#6 knew that positions 3 and 4 were different in the 3/4 and 4/3 fractions, but he could not understand each number's role (numerator and denominator) correctly.

In other cases, decimal numbers may be present to display other symbols of mixed numbers (Tian & Siegler, 2018). Still, even if students succeed in doing this, there is no guarantee that they understand the relationship between the two symbols, especially if they find them using a calculator or division operation. This condition is by the participant' answer #12 (see Table 5).

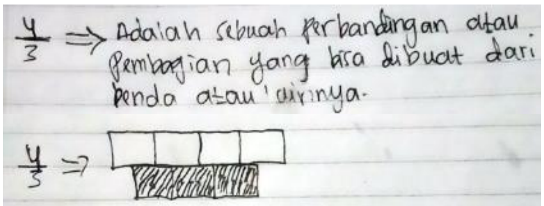
*Table 5. Sample Incorrect Answers in Interpreting and Representing the M4*

<table>
<tr><th>No</th><th>Figure</th><th>Explanation</th></tr>
<tr><td>1</td><td>

(Participant #12)</td><td>The 4/3 interpretation is connected with decimal numbers (generated by a calculator), although it seems meaningless. The participants could not depict 4/3 because they could not see the relationship between the fraction and the decimal they received.</td></tr>
</table>

*(Source: primary data, contribution of participant #12)*

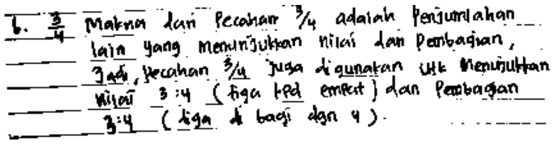
Furthermore, the interpretation of fractions as quotients and ratios also appears to be used by some participants in interpreting fractions (see Figure 1). The interpretation of fractions as quotients is used more often (6% in M1, 16% in M2, 3% in M3, and 16% in M4) than ratio (22% in M2). An interesting finding in this session was that some participants were able to recognize that a fraction is a division of two numbers (or as a quotient) only because the symbol " / " or "per" indicates division. This condition implies that even though the participants can illustrate 3/4 as three divided by four or three apples divided by four people, some failed to represent it in the correct visual model (see figure 5b).




*"is a comparison or division that can be made of one thing or another"*

*Figure 5a. Sample Participant's Correct Interpretation (in quotient), but Wrong Representation.*

*(Source: contribution of participant#34)*



*The fraction is also used to show the value of 3:4 (three by 4) and the division of 3:4 (three divided by 4)*

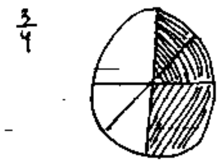


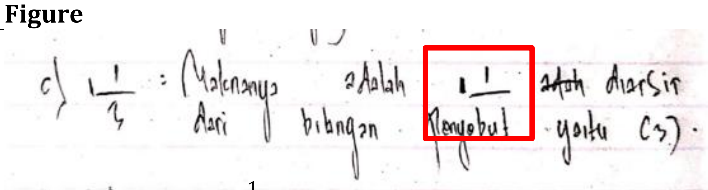
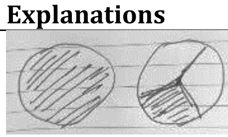
*Figure 5b. Sample Participant's Correct Interpretation (in ratio or quotient), but Wrong Representation.*

*(Source: contribution of participant#35)*

Understanding fractions as ratios is also still an obstacle for participants. Figures 5a and 5b show how students struggle to explain fractions as ratios. The participants viewed 4/3 as four boxes compared with three boxes or three slices of pie chart compared with four slices. In that way, they tried to use part-to-part ratios to make sense of part-to-whole fractions. These part-to-part ratios cannot be fractions because the ratio does not name a rational number; instead, it presents a comparison of two numbers (Chapin & Johnson, 2006). The knowledge that all fractions are ratios, but all ratios are not fractions (Kieren, 1980) should be imparted to students. Three-fourth (3/4) of a floor surface has a very different meaning than comparing the number of girls and boys in a class.

Understanding concepts is critical in learning mathematics (Viseu et al., 2020). For students prepared to be future teachers, mastery of concepts will affect their professional knowledge. No matter what kinds of issues they face, students will not be led astray if they have an excellent concept. One of the findings in this study shows that although students often struggle to describe the meaning of fractions in their language, a good understanding of the concept will lead them to the right rule (see Table 6). We found an interpretation that was slightly more interesting than the others. Table 6 shows the participant' struggle to make sense of the mixed number by explaining. This encouraged us to dig further through the interview and determine what the participant was thinking.

*Table 6. Sample of Wrong Explanation but With Right Representation*

| No | Figure | Explanations |
|---|---|---|
| 1. |  (i) *"The meaning is $1\frac{1}{.....}$ is shaded from a number that has a denominator of 3"* |  (ii) We received this illustration after interviewing the participant about the meaning of her responses. |

*(Source: Primary Data, Contribution of Participant #94)*

*Interview with Participant #94:*

*R : "I saw your response when giving the meaning of $1\frac{1}{3}$. What do you mean by writing this symbol $1\frac{1}{}$?*

*P94 : "I find it difficult to explain what the mixed numbers mean. That's why I wrote that symbol."*

*R : "ok, that's ok, but what does this mean?"*

*P94 : "so ..like this ma'am, this one is the full one" (she tries to shade the full circle)*

*"And this one" (she points to one as the numerator)*

*"We took only one part out of three."*

*R : "ok, I see once you explain it, but what if you rewrite what you just explained?"*

*P94 : "It's really hard for me to describe it, ma'am, but I'll try ....."!*

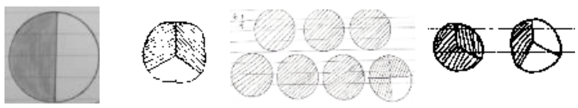
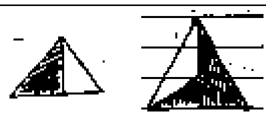
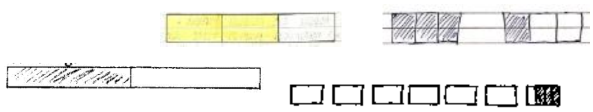
In the interview session, we found that Participant#94 knew the meaning of $1\frac{1}{3}$, but she could not give a clear explanation. Her idea of mixed numbers was illustrated by drawings a pie chat model (see Table 6ii), which described the problem she generated "*Ani has eaten one apple and then she ate 1/3 apple, how many apples has Ani eaten?*". This condition showed that asking students to represent their knowledge of fraction symbols in some models is one of the best ways to capture their understanding.

*Student' Knowledge on Representing Fractions*

Representation is something that cannot be separated in the learning process of fractions (Chapin & Johnson, 2006; Lamon, 2020). Interpretation and representation are two things that are interrelated and must be well understood by teachers when teaching various forms of fractions. Fractions can be represented in various forms, both in symbols and visual models. However, visual models in regional or geometric models, number lines, and sets of objects in representing fractions have a crucial position, especially at the elementary school level (Westenskow et al., 2014). The characteristics of elementary school students who still need concrete experience in understanding mathematics require teachers at this level to use concrete and visual representations related to mathematics, especially in learning fractions. This situation then becomes one of the reasons why teacher students' understanding of representing fractions is essential to be explored.

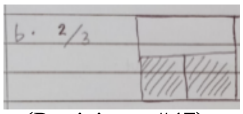
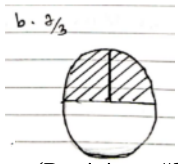
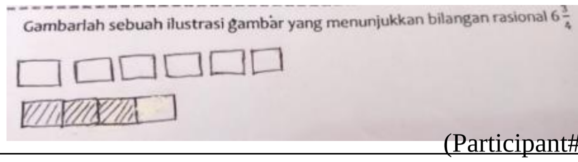
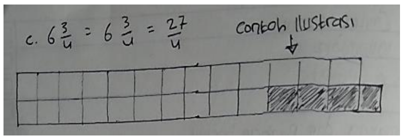
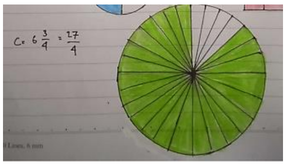
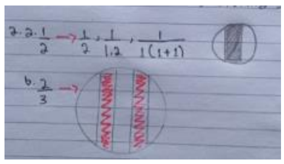
The findings show the participants'' tendency to use region models and their predominant dependency on a few types of models (e.g., area model) in showing the fractions symbol. Table 7 presents the frequencies of participants' correct models categorized as area, length, and set models. In particular, the participants employed fraction representation in the set of object models (1,4%) as they interpreted fractions, which is rare.

*Table 7. Proposed Correct Models by Students in Task D1-D4*

| No | Types of Models | Features of Model & Percentage of | | | | Example of |
|---|---|---|---|---|---|---|
| | | D1 | D2 | D3 | D4 | D1 / D2 / D3 / D4 |
| 1. | Area Model | Pie chart | | | |  |
| | | 41,6 | 29,6 | 33,3 | 19,1 | |
| | | Rectangular arrays | | | |  |
| | | 42,7 | 3.7 | 46,7 | 19,1 | |
| | | Triangle wedges | | | |  |
| | | 7,3 | 8,6 | - | - | |
| 2. | Length/ Linear Model | Vertical strips | | | |  |
| | | 3,1 | 1,23 | - | 4,7 | |
| | | Horizontal strips | | | |  |
| | | 4,17 | 56,79 | 20,0 | 57,1 | |
| 3. | Set of object Model | Set of Rectangular | | | |  |
| | | 1,04 | - | - | - | |

In this section, we also discuss the participants' mistakes in doing representation. We categorize these difficulties into three group, as presented in Table 8. In general, all these errors are related to struggles with part-whole understanding. According to Westenskow et al. (2014), if this error comes up, learners will undoubtedly have difficulties in handling fractions problems, one of which will be the inability to compare fractions.

*Table 8. Some of the Errors in Representing Fractions*

| No | Type of error | Explanation | Sample of representations |
|---|---|---|---|
| 1. | The sizes of parts are not the same | Ignore if b (denominator) is equal-size parts into which the whole has been divided |  (Participant #47)  (Participant#3)  (Participant #112) |
| 2. | Unequal size of unit/whole | Ignore if the unit or whole to be divided must have the same size and shape |   (Participant#12) |
| 3. | It does not conform to the fraction symbol | the representation presented does not match the fraction symbol |   (Participant#93)  (Participant#54)  (Participant#51) |

More than 99% of the participants represented all proper and improper fractions and mixed numbers in the area/region model, and only 1% used a set of objects (see Figure 2b, participant #42). The degree of this percentage is, of course, determined by the participants' perception of fractions. Practically, all of them seem familiar in a part-whole relationship compared to others.

*Implications for Teaching of Fractions*

This study addresses the question: How can we increase the understanding of undergraduate students on the topic of fractions, especially in terms of interpreting and representing fraction? Referring to NCTM standard and the Indonesian curriculum, fractions are taught in primary and secondary schools (Indonesia Ministry of National Education and Culture, 2016; NCTM, 2015). Therefore, improvements on these two topics should be made at the university and school levels.

Fraction learning, which is dominated by computational aspects rather than conceptual understanding, must be synchronized. Understanding fractions as a number and numeral (Albert B. Bennett et al., 2012; Musser et al., 2011; Siegler & Braithwaite, 2017), and knowing of what it is not and when it does not apply to fractions should emphasize the learning process (Lamon, 2020). The interrelationships between the five interpretations of fractions should be explored further in learning than presenting them individually (Chapin & Johnson, 2006). Not only that, the use of a rich context in the learning process of fractions is believed to be able to help students in understanding different interpretations of fractions and also in developing proportional reasoning, as claimed by several previous studies (Johar et al., 2018; Lamon, 2020; Wahyu et al., 2020). However, the compatibility between the given context with the symbol and the illustration of the fraction must be carefully considered.

The use of various manipulative tools in learning fractions is believed to be very helpful in clarifying the meaning of fractions and giving ideas about the various kinds of representations (Lamon, 2020; Mastuti, 2017). Especially at the elementary school level, students still need concrete experience in learning mathematics (Novita & Herman, 2021). Teachers can innovate by creating and using various manipulative tools in learning, especially technology-based ones. This advice is relevant for classroom teachers at this level. A similar study could be conducted to identify whether mathematics teachers and/or students trained as teachers can utilize various manipulative tools in teaching fractions.

Furthermore, the contributions of this research to the literature are: (1) to provide an overview of the second-year university students' knowledge in interpreting and representing fractions; (2) the features and issues that the second-year university students face in studying fractions are revealed; (3) the research results can be used to develop ideas for designing fraction learning in primary teacher education programs and justify focusing on developing conceptual understanding rather than computational aspects. This recommendation also opens up opportunities for further research.

## Conclusion

Interpretation and representation are two significant aspects in understanding fractions because they are interrelated. Based on the findings, it can be concluded that student teachers' understanding of fractions is still limited and they face some challenges in interpreting and representing them. The interpretation of the relationship part-whole (equal-size parts) is the most frequently used by the participants compared to the parts-whole relationship as equal-size groups and the other four interpretations. However, there are still many critical errors in the parts-whole relationship interpretation, such as ignoring parts as "equal parts," failing to understand the meaning of numerator and denominator, and disregarding the size or quantity of the whole area or set. Furthermore, the understanding of fractions, which is only restricted to part-whole relationships, still leaves a gap for difficulties in understanding the interpretation and representation of the improper fraction symbol. We found that participants who succeeded in interpreting and representing proper fractions were not necessarily able to interpret improper fractions. In addition, completely misunderstanding fractions as ratios is another issue that requires attention.

On the other hand, related to fraction representation, students who can express fraction symbols in real-world situations using verbal words cannot necessarily represent them in pictures or models correctly. Therefore, representation in some models is essential to identify the meaning of fractions and see how students understand them.

## Recommendations

Various efforts described in the implications section of this research need to be carried out on the primary school teacher education program. Furthermore, the findings can be used to investigate in greater depth the difficulties of students in representing fractions as a measure (in number line) and operators. Future research can conduct some planning learning activities that give student teachers more about the diverse interpretation of fractions. The research findings also suggest using the various models in teaching fraction.

## Limitations

This study had two significant limitations. First, while the finding of this study determines the student teachers' understanding and paints a realistic picture of the difficulties possessed them related fraction during their second year in primary teacher education program, caution should be exercised in using them for generalization. Keep in mind that this study was conducted exclusively in two private universities. Thus, a counterpart of this study may also be delved into, considering the public university. Second, this study only focuses on the structure of fractions related to their meaning and does not cover operations with fractions.

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### Authorship Contribution Statement

Novita: Conceptualization, design, data acquisition, analysis, writing, final approval, critical revision of manuscript. Herman: Design, data acquisition, Editing/reviewing, critical revision of manuscript, supervision. Dasari: Data acquisition, data analysis / interpretation, statistical analysis. Putra: Technical and material support, data acquisition, observer, interviewer and drafting manuscript.

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