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Examining the Conceptual and Procedural Knowledge of Decimal Numbers in Sixth-Grade Elementary School Students

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Abstract: In this article, we present the results of empirical research using a combination of quantitative and qualitative methodology, in which we examined the achievements and difficulties of sixth-grade Slovenian primary school students in decimal numbers at the conceptual and procedural knowledge level. The achievements of the students ($N = 100$) showed that they statistically significantly ($z = -7{,}53$, $p < .001$) better mastered procedural knowledge ($M = 0.60$, $SD = 0.22$) than conceptual knowledge ($M = 0.37$, $SD = 0.17$) of decimal numbers. Difficulties are related to both procedural and conceptual knowledge, but significantly more students have difficulties at the level of conceptual knowledge. At the level of procedural knowledge, or in the execution of arithmetic operations with decimal numbers, we observed difficulties in transforming text notation into numerical expressions, difficulties in placing the decimal point in multiplication and division, and insufficient automation of mathematical operations with decimal numbers. At the level of conceptual knowledge of decimal numbers, the results indicate difficulties for students in understanding the place values of decimal numbers, in estimating the sum, product and quotient of decimals with reflection and in mathematical justification. In relation to difficulties in justification, we observed an insufficient understanding of the size relationship between decimal numbers and difficulties in expressing them in mathematical language. The results indicate that to overcome such difficulties in the learning and teaching of mathematics, more balance between procedural and conceptual knowledge is needed.

Keywords: *Conceptual knowledge, decimal numbers, math learning difficulties, procedural knowledge.*

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Introduction

Mathematics education is designed to build concepts and connections, to learn and teach procedures that enable individuals to integrate into a system of (mathematical) ideas and, consequently, to integrate into the culture in which we live (Žakelj et al., 2011). Theories and classifications of knowledge can help us to plan activities for the classroom, to identify cognitive skills, to plan tasks for testing and assessment, to grade tasks according to taxonomic levels (tasks of varying difficulty), and to interpret learners' results.

Although it is not always possible to separate basic and conceptual, and procedural knowledge, it is useful to distinguish between the two types of knowledge in order to better understand the development of knowledge. Conceptual knowledge enables the understanding of concepts and ideas that define the domain of relationships, while procedural knowledge, including skills, algorithms and strategies, enables quick and efficient problem-solving (Dijanić et al., 2015).

Slovenian students' knowledge of numbers, especially decimals, has been a persistent area of concern for many years, even though the performance of Slovenian students in Grade 8 mathematics in international surveys (e.g., TIMSS and PISA) is above the international average. Based on a literature review, we have prepared a study to analyse in more detail the difficulties encountered by Slovenian students in dealing with decimals. Research shows that students' proficiency in basic and conceptual knowledge is lower than their proficiency in procedural knowledge, not only in the areas of decimals but also in other content areas of mathematics (Lawson, 2007; Lortie-Forgues & Siegler, 2017). With this reason, the research aimed firstly, to investigate whether there are differences between students' achievement in decimals at the level of basic and conceptual knowledge and procedural knowledge, and secondly, what the potential difficulties students

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may have in different basic arithmetic operations of decimals at the level of basic and conceptual knowledge and procedural knowledge.

Basic and Conceptual Knowledge

According to Isleyen and Işik (2003), conceptual knowledge in mathematics refers to the comprehension of mathematical concepts, rules, and propositions by means of symbols and demonstrations. This form of knowledge establishes connections between various mathematical notions, hence facilitating a full comprehension of mathematical concepts. In a similar vein, the authors Hiebert and Lefevre (1986) expounded upon the notion of conceptual knowledge, characterising it as a form of knowledge that encompasses an understanding of concepts and principles, as well as the interconnections between them.

Knowledge of concepts, or conceptual knowledge, is concerned with generalizing and abstracting from specific examples. Knowing mathematical definitions and rules is not the same as having a conceptual understanding. Though it is impossible to say that students have conceptual comprehension, they might be able to recollect certain definitions, guidelines, and practices. The student must be able to explain why a mathematical statement is true or the origin of a mathematical rule in order to show conceptual comprehension (Zuya, 2017).

Procedural Knowledge

According to Canobi (2009) and Rittle-Johnson et al. (2001), procedural knowledge can be defined as the ability to possess and effectively utilise various skills, strategies, algorithms, measures, and procedures in order to accomplish diverse objectives. According to Hiebert and Lefevre (1986), procedural knowledge encompasses the use of formal language, such as mathematical symbols, rules, algorithms, procedures, or instructions, which are executed in a sequential manner. The main characteristic between all of these procedures is their adherence to a predetermined sequential order.

Flexibility is an important aspect of procedural knowledge, as it pertains to an individual's ability to identify and select the most optimal processes or procedures within a particular setting. According to Star (2005), procedural knowledge encompasses not only the execution of procedures but also the ability to appropriately and adaptively select a procedure. The concept of cognitive flexibility refers to an individual's capacity to adopt several viewpoints and modify cognitive processes, hence enabling the generation of alternative problem-solving strategies in the face of difficulties or cognitive barriers (Krutetskii, 1976; Leikin, 2007). An individual exhibiting a notable level of adaptability will modify his or her mental pathways or problem-solving strategies in situations in which they prove to be unproductive and fail to yield a resolution.

RQ1: Are there differences between students' achievements in decimal numbers on basic and conceptual knowledge and procedural knowledge?

RQ2: What are possible students' difficulties in different basic arithmetic operations of decimal numbers at the basic and conceptual knowledge (estimate and justify the estimate) and procedural knowledge (written calculations)?

Literature Review

Research conducted on numbers and mathematical operations has revealed a prevalent issue of poor academic performance among elementary school children across many nations. For instance, empirical investigations such as the Trends in International Mathematics and Science Study (TIMSS) and the Programme for International Student Assessment (PISA) have provided evidence indicating that a significant proportion of primary school students worldwide exhibit poor achievement levels in the domain of numerical proficiency (Mullis et al., 2016; The Organization for Economic Cooperation and Development [OECD], 2019).

In a study conducted by Purnomo et al. (2014), a group of 80 sixth-grade students from Indonesia were examined to evaluate their proficiency in three distinct areas: (a) comprehension of numerical meanings, (b) comprehension of mathematical operations, and (c) comprehension of the application of numbers and operations inside computational systems. The study revealed that a majority of students obtained low scores across all three components. According to Purnomo et al. (2014), the underperformance of students can be attributed to two key problems. Firstly, both teachers and students heavily rely on procedural knowledge, particularly standard computing algorithms. Secondly, learning resources, such as textbooks, predominantly emphasise textual rules and algorithms.

Misconceptions, as identified by Merenluoto and Lehtinen (2002), Yang and Sianturi (2019a, 2019b), and McNeil and Alibali (2005), often show resistance to change and have the potential to become deeply ingrained. According to Merenluoto and Lehtinen (2002), the presence of misconceptions among students can impede their capacity to generate precise solutions to mathematical problems and provide sound reasoning in support of those solutions. According to Yang and Sianturi (2019a, 2019b), it is important to note that misconceptions have the potential to hinder the development and utilization of effective and efficient problem-solving strategies. Therefore, it is crucial to identify and address misconceptions, as they can significantly impact the conceptual understanding of the mathematics of elementary students.

The study conducted by Sianturi et al. (2023) examined the comprehension of numerical concepts and mathematical operations among a sample of 372 students in the fifth grade. The results indicated that a significant proportion of students demonstrated a proficient comprehension of the fundamental concepts related to numbers and mathematical operations. The assessment of their ability to evaluate the reasonableness of computational results indicated the lowest level of proficiency. It is noteworthy that, despite the students' low scores, a significant proportion of them exhibited a high level of confidence in the responses they provided.

Mastery of decimals is not only important for mathematics but also has an impact on a student's success in physics, biology, chemistry, etc., and in many other areas (Lortie-Forgues et al., 2015). It also has an impact on a student's success in later life, outside the school system, in professional activities where knowledge of decimal numbers is needed, such as in pharmacy (measuring quantities of drugs), in daily life (e.g., using recipes in the kitchen, understanding food label declarations), and knowledge of decimal numbers is important for understanding statistical data (Lortie-Forgues & Siegler, 2017).

However, mastering decimal numerals is a difficult challenge for numerous students. Complete comprehension requires a distinct cognitive approach to numbers, particularly multiplicative thinking. This kind of thinking is not inherently intuitive but rather requires a reevaluation of the numerical relationship, which diverges from the cognitive framework required for additive relationships (Forman et al., 1998).

Several studies have indicated that students as well as adults had limited comprehension of decimal numbers (Lai & Tsang, 2009; Moloney & Stacey, 1997; Sengul & Gulbagci, 2012; Steinle, 2004). Additionally, teachers may encounter challenges in comprehending and learning decimals (Girit & Akyuz, 2016). According to the findings of Lai and Tsang (2009), students possess adequate procedural knowledge regarding decimal numbers, although show limited conceptual knowledge and comprehension of decimal notation. There is increasing evidence that students may become proficient in performing mathematical procedures, but struggle with understanding the underlying concepts. D'Ambrosio and Kastberg (2012) also point out a major problem of conceptual knowledge in the area of decimal numbers, as did Sadi (2007), who states that students have more difficulties understanding decimal numbers due to difficulties in conceptual knowledge relative to other types of numbers.

In their study, Linchevski and Livneh (1999) examined the phenomenon of students frequently misapplying the order of operations, attributing it to a lack of comprehension regarding the meaning of operational symbols. In a similar vein, Sadi (2007) documented that a considerable number of students encounter difficulties when comparing fractions. One common misconception observed is the erroneous belief that 3/8 is smaller than 2/5, solely based on the denominators.

Further research (Purnomo et al., 2014; Yang & Sianturi, 2019a, 2019b) has revealed challenges encountered by students when comparing fractions. These issues arise from their dependence on rule-based methods, such as the emphasis on identifying common denominators, as well as the misperception of treating fractions as whole numbers.

Moreover, a study conducted by Vamvakoussi and Vosniadou (2010) revealed a prevalent misperception among students, namely the belief that a decimal with a greater number of digits indicates a larger value. For example, individuals may erroneously assume that 0.23 possesses a greater magnitude than 0.9.

Hiebert and Wearne (1986) found that elementary school students have difficulty with decimal numbers that include one or more "0" immediately to the right of the decimal point; many ignore these "0"s and, for example, claim that 0.02 · 0.03 = 0.6. Bell et al. (1981) also researched students' difficulties with decimal numbers and found that students consider decimals to be "below zero" or negative numbers. Place-value columns include "oneths" to the right of the decimal point. One hundredth is written as 0.100, and 1/4 can be written either as 0.4 or as 0.25. Irwin (2001), in her study "Using everyday knowledge of decimals to improve understanding", also identified some of the difficulties that students have with decimal numbers. She highlighted the following misunderstandings: »Longer decimal fractions are necessarily larger. Putting a zero at the end of a decimal number makes it ten times as large« (Irwin, 2001, p. 402).

A study conducted by Lortie-Forgues and Siegler (2017) revealed a particular form of misunderstanding of decimals. The researchers delved deeper into the conceptual understanding of decimals and found that the students in the study excelled at estimating the size relationships between decimals and performing written computational algorithms. However, they had great difficulty predicting the value of the product or quotient when two decimal numbers were involved, especially when both decimals were less than 1. Most students in the study incorrectly believed that the product of two positive decimals less than 1 was greater than each of them separately and that the quotient, when divided by a decimal less than 1, was less than the original quotient.

The fact that students' knowledge is lower at the level of basic and conceptual knowledge than at the level of procedural knowledge is not only true for decimals. Research shows very similar findings for other content areas of mathematics: Lawson (2007) points out that students frequently encounter difficulties when it relates to grasping conceptual comprehension. RQ1: Are there differences between students' achievements in decimal numbers on basic and conceptual knowledge and procedural knowledge?

RQ2: What are possible students' difficulties in different basic arithmetic operations of decimal numbers at the basic and conceptual knowledge (estimate and justify the estimate) and procedural knowledge (written calculations)?

To make the content as accessible as possible to students, the learning and teaching of mathematics is based on the fact that mathematics is a subject where the knowledge is continually being built on, which means that more complex knowledge builds on the basic knowledge that the individual has already acquired or is expected to have acquired. Therefore, when introducing new concepts and content, we take into account what the learner's prior knowledge is, how it is applied and to what extent arithmetic facts and procedures are automated. Any help is ineffective if the student is consolidating current content but has not acquired the basic knowledge and skills (e.g., practising multiplication and division techniques is of no use if the student is not good at addition (Jelenc & Novljan, 2001)). Likewise, Doerr (2006) emphasises the importance of instructors being capable of predicting the student's progression from prior knowledge to newly acquired knowledge. This indicates that, in the context of decimal number instructions, the teacher must be proficient in the relationships between decimal numbers and the base ten structure acquired through integer reasoning, as well as the measure and part-whole sub-constructs of fractions and equivalent fractions. There are numerous connections to be made, and such connections can take various forms within the classroom setting (Takker & Subramaniam, 2019).

Researchers emphasise the criticality of instructing for conceptual comprehension in order to enhance students' performance in conceptual knowledge (Lawson, 2007; Protheroe, 2007). It is necessary to teach students the concepts underlying procedures (National Council of Teachers of Mathematics [NCTM], 2000). Hiebert (1999) explains that students are frequently unmotivated to learn the underlying concepts that support a procedure once they have mastered it. Therefore, it is advised by the NCTM (2000) that students acquire knowledge of the underlying concepts prior to or during the instruction of procedures, rather than subsequent to it.

Bonotto (2001) argues that students' challenges with decimal numbers are rooted in teachers' teaching strategies that lack relevance to the students' real-world situations. Students frequently confront decimal numbers in the form of some stereotype word problems. According to Niss et al. (2007), word problems have been a part of school mathematics curricula for centuries and are presented as mathematical applications; however, they are essentially pure mathematics problems disguised as words.

Similarly, Hiebert and Lefevre (1986) emphasize that to develop conceptual knowledge of decimals, students need to relate them to real-life situations. Irwin (2001) also examined the role of context related real-life situations in the development of knowledge of decimals. Students solved tasks concerning prevalent misunderstandings concerning decimals. Half of the pairs solved problems in familiar real-life context and half solved problems that were presented without context. The analysis of the pre-test and post-test outcomes revealed that the students who engaged with contextual problems demonstrated substantially greater improvements in their knowledge compared to those who tackled problems devoid of any contextual elements. The author states that the students who worked on the contextual problems built a scientific understanding of decimals through reflections on their real-life everyday experiences concerning the meaning of decimals and the outcomes that result from decimal computations.

Methodology

Problem Definition and Aim of the Study

According to the results of international surveys TIMSS 2015 (Mullis et al., 2016), PISA 2018 (OECD, 2019) and national external examination of mathematical knowledge (Republiški Izpitni Center [Slovenian National Examinations Centre], 2018, 2019), the performance of Slovenian students in grade 8 in mathematics is above the international average, but a consistent area of concern has been observed in their proficiency with numbers, particularly decimals, which has yielded comparatively lower scores over the years. Understanding multiplication and division of decimals is also weak in countries that are top performers on international comparisons of mathematical achievement, for example, China (Liu et al., 2014; OECD, 2014).

Focusing upon the outcomes of the national and international studies and the situation regarding Slovenian students' knowledge of decimals, our study aimed to investigate the academic achievement of primary school students in the sixth grade within the domain of decimal numbers, with a specific focus on their conceptual and procedural knowledge. At the level of conceptual knowledge, our research focused on the ability to estimate and justify the values of computational operations between decimal numbers before performing written algorithms and compared this achievement with the achievement in performing written computational operations (procedural knowledge). The aim of our study was also to examine possible differences in student achievements in terms of conceptual and procedural knowledge, with a focus on identifying differences in the achievement of estimating and justifying the value of a numerical expression with the achievement in written computation. Additionally, we aimed to identify the difficulties encountered by students in acquiring decimal number knowledge at both the conceptual and procedural levels.

Research Questions

According to the aim of the study we formulated the following research questions:

RQ1: Are there differences between students' achievements in decimal numbers on basic and conceptual knowledge and procedural knowledge?

RQ2: What are possible students' difficulties in different basic arithmetic operations of decimal numbers at the basic and conceptual knowledge (estimate and justify the estimate) and procedural knowledge (written calculations)?

Sample and Data Collection

The study included a sample size of one hundred children in the 6th grade from six randomly selected Slovenian schools, which represents approximately 0.5 % of all Slovenian 6th graders. Six math teachers with the same level of education (university degree) and at least 10 years of work experience participated in the study. Data were collected using a 20-task instrument: 10 tasks required conceptual knowledge (Test 1) and 10 tasks required knowledge of procedures on decimal numbers (Test 2). We designed these tests in collaboration with experts in mathematics didactics in line with the curriculum standards of the Slovenian math curriculum as mandated by the Ministry of Education. Some of the items used in this study were adapted from reviews of the literature and previous studies, while others were developed by the researcher.

Test 1

Test 1 (T1) tasks assess the procedural knowledge of decimal numbers: written calculations as addition, multiplication and division of decimal numbers—convert a mathematical text (verbal task) into a numerical expression; calculate the value of a numerical expression with decimals, taking into account the order of mathematical operations; formulate the answer; and know and use symbolic notation among decimal numbers (<, >, =). Identify parts of a whole in an image, write them as a decimal number and draw a part of a whole in a given image. Solve simple verbal tasks—independently choose a solution strategy, infer from a unit to a set, express the result with a decimal number and a measure unit, read data from a bar chart and represent them in a table.

Test 2

Test 2 (T2) assesses the conceptual knowledge of decimal numbers: without written calculations (addition, multiplication and division), evaluate the sum, product and quotient of decimal numbers; justify the estimates using mathematical facts; infer; and understand and decode symbolic notation among decimal numbers (<, >, =). Recognize parts of a whole in an image, compare their sizes and justify the answer. Convert measure units. Infer from a set to a unit and from a unit to a set. Solve a verbal task.

Our research followed ethical procedures. Consent from the parents for the students' participation in the research was obtained beforehand. Students and their parents were briefed on the objectives of the study and the protocols for data processing. Confidentiality measures were consistently implemented and all data were anonymised to protect students' privacy. Transparency was ensured through proactive communication with all stakeholders, ensuring that all stakeholders were informed about the research process.

Analyzing of Data

The data collected with both tests represent the dependent variables in the statistical context. For each measuring instrument, we analysed the objectivity, reliability and validity. The objectivity of tests was considered regarding testing, evaluation of results and interpretation of test results. The internal consistency (reliability) was verified using Cronbach's alpha coefficient measures, and it was determined that reliability was very high in the case of procedural knowledge (α_{proc} = 0.889); for conceptual knowledge, it was (α_{con} = 0.700). To verify the validity of both tests we used factor analysis. Data collection was performed according to the same procedure for both tests. Test 2 was conducted one week after Test 1 under the same conditions and with the same tester. The results were evaluated uniformly in accordance with predetermined criteria for all the tested students, thus ensuring the objectivity of both tests. We conducted and discussed data analysis of both tests in the research group and with peers. Individual items on the test were computed according to the level of knowledge—conceptual knowledge and procedural knowledge—and basic descriptive statistics were calculated.

Statistical Processing

For statistical processing of the quantitative data, we used the Statistical Package for Social Sciences (SPSS). To determine the significance of the differences between conceptual and procedural levels of knowledge we used the Wilcoxon signed-rank test, a nonparametric equivalent of the parametric t-test, as assuming the homogeneity of the variances was unjustified, due to the abnormal distribution. We calculated the effect size using the formula: $r = \frac{Z}{\sqrt{N}}$, for which $|r|$ = 0.1

is a small effect size, $|r|$ = 0.3 is a medium effect size and $|r|$ = 0.5 is a large effect size (Fritz et al., 2012). To calculate the correlation between procedural and conceptual knowledge we used Spearman's correlation.

Results

In the continuation, we will present the results of research of 6th-grade elementary school students in decimal numbers. The initial focus is on the students' achievements in both procedural and conceptual knowledge, followed by an exploration of their difficulties in these areas.

Students' Achievements in Procedural and Conceptual Knowledge

RQ1: Are there differences between students' achievements in decimal numbers on basic and conceptual knowledge and procedural knowledge?

Table 1. Descriptive Statistics for Procedural Knowledge and Conceptual Knowledge and the Result of Related-Samples Wilcoxon Signed Rank Test for the Differences Between Levels of Knowledge

	N	Min	Max	*M*	*SD*	*SE*	*z*	*p*
procedural knowledge	100	0.10	1.00	0.60	0.22	0.02	-7,53	<.001
conceptual knowledge	100	0.08	0.85	0.37	0.17	0.02		

N – number of students, *M* – mean, *SD* – standard deviation, *SE* – standard error of a mean, Min – minimum, Max – maximum, *z* - value of the *Related-Samples Wilcoxon Signed Rank Test*, *p* – level of statistical significance

The results (Table 1) show that students have statistically significantly (z = -7,53, $p < .001$) better procedural knowledge (M = 0.60, SD = 0.22) than conceptual knowledge (M = 0.37, SD = 0.17) of decimals. The correlation coefficient (r = .492; $p < .001$) shows a significant correlation between procedural and conceptual knowledge. Individually selected tasks from Test 1 and Test 2 were checked as pairs of content comparable goals (Table 2) but at different levels (Test 1 tasks on a procedural level; Test 2 tasks on a conceptual level) – for example, task T16 of Test 1 and task T26 of Test 2.

Table 2. Descriptive Statistic for Individual Tasks of Test 1 and Test 2 According to Procedural Knowledge and Conceptual Knowledge

Var.	Description	*N*	*M*	*SD*	*SE*	Min	Max
T11P	Calculate the square of a decimal number with written multiplication.	100	0.69	0.39	0.04	0.00	1.00
T21C	Estimate the square of a decimal number without written multiplication.	100	0.12	0.33	0.03	0.00	1.00
T12P	Use written addition to calculate the sum of two decimals and the difference of a decimal and an integer. Write the answer.	100	0.81	0.23	0.02	0.25	1.00
T22C	Estimate the sum of decimals without written addition and estimate the difference between a decimal and an integer and justify the estimate.	100	0.43	0.27	0.03	0.00	1.00
T13P	Calculate the product of two decimal numbers. Write the answer.	100	0.63	0.34	0.03	0.00	1.00
T23C	Estimate the product of decimals without written multiplication. Justify the estimate.	100	0.21	0.33	0.03	0.00	1.00
T14P	Calculate in writing the product of two decimal numbers. Calculate in writing the difference between a decimal and an integer. Write the answer.	100	0.66	0.33	0.03	0.00	1.00
T24C	Compare the estimate of the product of decimal numbers (without written multiplication) with the number 1,000 and justify the answer.	100	0.25	0.29	0.03	0.00	1.00
T15P	Calculate in writing the difference between two products of decimals. Write the answer.	100	0.70	0.38	0.04	0.00	1.00
T25C	Estimate the values of two products (whole and decimal numbers) and compare them in terms of size. Justify the estimate.	100	0.45	0.37	0.04	0.00	1.00
T16P	Calculate in writing the difference between two quotients (integer and decimal). Write down the answer.	100	0.42	0.42	0.04	0.00	1.00

Table 3. Continued

Var.	Description	*N*	*M*	*SD*	*SE*	Min	Max
T26C	Estimate the values of two quotients (integer and decimal) and sort them by size. Justify the estimate.	100	0.26	0.28	0.03	0.00	1.00
T17P	On the image, recognize the part of the whole and write it down as a decimal number. Colour the part of the whole.	100	0.67	0.28	0.03	0.00	1.00
T27C	On the image, identify the parts of the whole and compare them in terms of size. Justify the answer.	100	0.75	0.26	0.03	0.00	1.00
T18P	Calculate the value of a numerical expression with decimal numbers, taking into account the order of mathematical operations.	100	0.52	0.34	0.03	0.00	1.00
T28C	Estimate the sizes of the quotients and compare them in order of size (the dividend is an integer; the divisors are decimal numbers between 0 and 1) and justify the estimates.	100	0.30	0.34	0.03	0.00	1.00
T19P	To infer from singular to plural. Independently choose a solving strategy (write the multiplication with decimals and calculate it). Write down the answer.	100	0.48	0.37	0.04	0.00	1.00
T29C	Convert units of measurement. Infer from plural to unit and from unit to plural.	100	0.31	0.44	0.04	0.00	1.00
T110P	Read the data from the diagram, write it in the table and add its values. Write the answer.	100	0.45	0.34	0.03	0.00	1.00
T210C	Read data from a graph and record it in a table. Draw conclusions. Choose an appropriate solution strategy. Write down the answer.	100	0.55	0.31	0.03	0.00	1.00

T1iP – procedural task i on Test 1, T2iC – conceptual task i on Test 2, $1 \leq i \leq 10$; *N* – number of students, *M* – mean, *SD* – standard deviation, *SE* – standard error of a mean, Min – minimum, Max – maximum

For tasks with content comparable goals (Task 2 to Task 6) we calculated the difference between students' achievements in individual tasks of Test 1 and Test 2. Students' achievements in tasks with content-comparable goals are significantly higher in Test 1 than in Test 2 (procedural knowledge) (Table 1).

Students' Difficulties on Procedural and Conceptual Knowledge

In the following sections, a qualitative analysis of the results of the individual tasks is conducted, in which we examine students' difficulties on procedural and conceptual knowledge. This leads to addressing the second Research Question (RQ2): What are possible students' difficulties in different basic arithmetic operations of decimal numbers at the basic and conceptual knowledge (estimate and justify the estimate) and procedural knowledge (written calculations)?

a) Addition of decimals

Examples of tasks:

- Task T12: Determine the sum of decimal numbers by written addition and compare decimals and integers in order of their size.
- Task T22: Estimate the sum of decimals by reflection, without written addition, and justify the estimate.

The performance on procedural task T12 (M = 0.81, SD = 0.23, *"Computationally check whether the inequality 8.29 + 0.99 > 10 holds true."*) is statistically significantly higher (z = -7.719, p < .001) than the performance on conceptual task T22 (M = 0.43, SD = 0.27, *"Without a written calculation, estimate whether the inequality 8.29 + 0.99 > 10 holds true. Justify your answer."*). In the written addition or subtraction of two decimal numbers, the students were mostly successful, but there were some difficulties in transforming the text into a numerical expression by writing the decimal numbers correctly (e.g., in the subtraction calculation, they write the larger number under the smaller one [Figure 1], and in the subtraction calculation, they write the whole part of decimal number under the decimal part [Figure 2]).

A lower performance than in T12 was observed in conceptual task T22, which required estimating the sum of two decimals by reflection and justifying the estimation without a written addition. We noted difficulties both in estimating the sum and in justifying this estimation. Figure 3 shows an example of a student who, when justifying his estimate of the size of the sum 8.29 + 0.99, only considered the whole part of the decimal number and ignored the tens and hundreds without explanation ("Because 8 + 0 = 8 and therefore not greater than 10."). The student (Figure 4), contrary to the task requirement, calculated the sum in writing instead of, according to the task requirement, estimating and justifying the sum without a written addition.

It is observed that the justification of the estimation is often incomplete, inadequate or incorrect (Figure 5), with inadequate or inappropriate vocabulary, and in large cases, there is a complete absence of justification. The observation that the justification of the estimation is frequently lacking in sufficiency, appropriateness or correctness (as depicted in Figure 5) is noteworthy.

Students who justified the task correctly first rounded the smaller decimal number to the nearest whole number and estimated that the sum was still less than 10. This suggests that students may already have difficulties with number representations, where they cannot transfer their routine procedural knowledge of rounding decimals, without understanding it, to a new situation that requires the application of their knowledge in a new context. This observation underscores a fundamental challenge in mathematics education: the ability to apply conceptual understanding to novel situations, rather than relying solely on procedural knowledge.

In many cases, students demonstrate proficiency in mathematical operations such as rounding decimals when the context is familiar and the task is routine. However, when they have to apply these skills in unfamiliar contexts or in tasks that require a higher level of conceptual understanding, such as estimating sums without explicit calculations, their understanding is superficial. This gap between procedural fluency and conceptual understanding is a critical area for educational intervention (Perry & Len-Ríos, 2019).

Table 4. Examples of Solved Tasks T1 and T2

<table>
<tr><th>Task T12 of Test T1</th><th>Task T22 of Test T2</th></tr>
<tr><td>

Difficulty transforming text notation into numeric expression

Incorrect choice of computational operation.
(Calculates the sum instead of the difference)

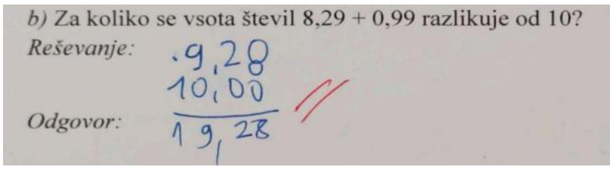


Figure 1. Example 1 of a Student's Difficulties with Procedural Knowledge

</td><td>

Difficulty justifying the estimate of the sum of two decimal numbers

When justifying, the student considers only the whole parts of the decimal number, leaving out the decimal part without explanation.

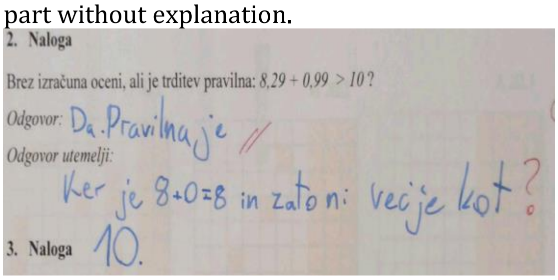


Figure 3. Example 1 of a Student's Difficulties with Conceptual Knowledge

</td></tr>
<tr><td>

Difficulty transforming text notation into numeric expression

Incorrect notation of a mathematical calculation (when writing a subtraction calculation, the student writes the larger number under the smaller number and the whole part of the decimal number under the decimal part).

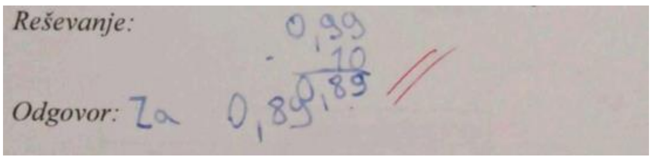


Figure 2. Example 2 of a Student's Difficulties with Procedural Knowledge

</td><td>

Difficulty justifying the estimate of the sum of two decimal numbers

The justification is based on a calculation.

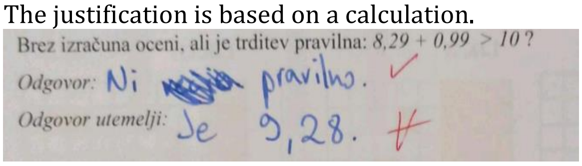


Figure 4. Example 2 of a Student's Difficulties with Conceptual Knowledge

Difficulty estimating the sum of two decimal numbers

An incorrect estimate of the sum of two decimal numbers.

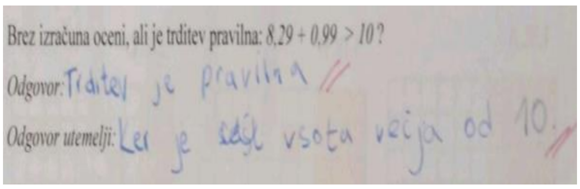


Figure 5. Example 3 of a Student's Difficulties with Conceptual Knowledge

</td></tr>
</table>

b) Multiplying a decimal greater than 1 by a decimal between 0 and 1

Examples of tasks:

- Task T13: Determine the product of decimals by written multiplication.

- Task T23: Estimate the product of decimals by reflection, without written multiplication, and justify the estimate.

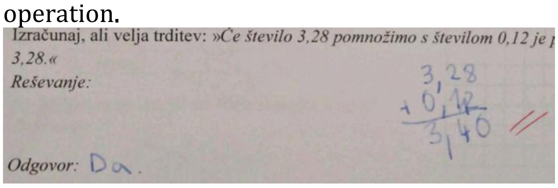
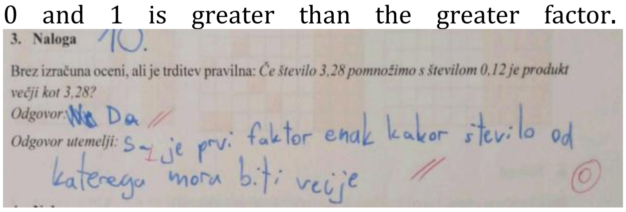
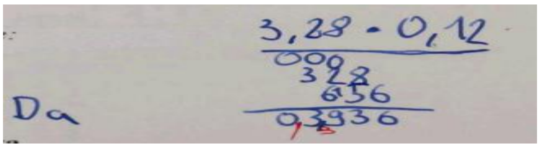
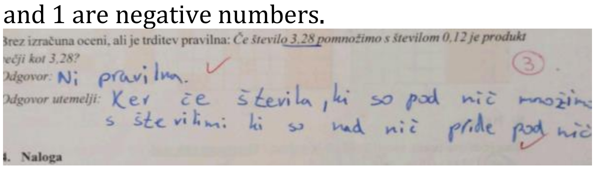
In addition, for the multiplication of decimals, the analysis of the performance results for T13 and T23 shows higher performance in written multiplication than in estimating the product by reflecting on and justifying the estimation. The performance of procedural task T13 (M = 0.63, SD = 0.34, *"Computationally check whether the statement 3.28 multiplied by 0.12 is greater than 3.28."*) is statistically significantly higher (z = -6.940, p < .001) than the performance of the conceptual task T23 (M = 0.21, SD = 0.33, *"Without written calculation, estimate whether the statement is correct: If the number 3.28 is multiplied by 0.12, the product is greater than 3.28. Justify your answer.").*

When multiplying a decimal greater than one by a decimal between zero and one in written form, students showed a slightly lower level of proficiency compared to the written addition of two decimal numbers. Similar to the written addition of decimal numbers, difficulties were observed in the transformation of text notation into numerical expressions (e.g., incorrect selection of arithmetic operation [Figure 6]) and in the calculation of numerical expression values (e.g., reduced reliability in executing the multiplication procedure, incorrect placement of the decimal point or inadequate understanding of the rules for multiplying decimal numbers [Figure 7]).

Performance in Task T23, where students estimated the size of the product of a decimal number greater than 1 by a decimal number between 0 and 1 with reflection and justified the estimate, was very low (M = 0.2). In the justification of the product estimate, the student (Figure 8) incorrectly assumed that the product of two positive decimal numbers is greater than the larger factor (3.28 · 0.12 > 3.28). An example of the solution provided by a student who wrongly assumed that decimal numbers between 0 and 1 are negative numbers is depicted in Figure 9. To support the claim that 3.28 · 0.12 > 3.28 is incorrect, the student (Figure 9) stated: "If the numbers below zero are multiplied by the numbers above zero, we get the numbers below zero."

In the multiplication of decimal numbers, the outstanding difficulty is justification, which is often inadequate, inappropriate and based on incorrect conceptual representations (e.g., decimal numbers between 0 and 1 are negative numbers), with many inconsistencies in expression. However, students often do not approach justification.

Table 4. Examples of Solved Tasks T1 and T2

Task T13 of Test T1	Task T23 of Test T2
Difficulty transforming text notation into numeric expression Inadequate selection of mathematical operation.  **Figure 6. Example 1 of a Student's Difficulties with Procedural Knowledge**	**Difficulty in the estimation of the product of two decimal numbers** The student thinks that the product of a decimal number greater than 1 and a decimal number between 0 and 1 is greater than the greater factor.  **Figure 8. Example 1 of a Student's Difficulties with Conceptual Knowledge**
Difficulty computing the value of a numerical expression with decimal numbers Incorrect placement of the decimal point. Lack of knowledge of the rules for multiplying decimal numbers.  **Figure 7. Example 2 of a Student's Difficulties with Procedural Knowledge**	**Difficulty justifying the estimation of the product of two decimal numbers** The student thinks that decimal numbers between 0 and 1 are negative numbers.  **Figure 9. Example 2 of a Student's Difficulties with Conceptual Knowledge**

c) Multiplication of two decimal numbers and comparison of the product with an integer

Examples of tasks:

- Task T14: Determine the product of two decimal numbers using written multiplication and compare the product (decimal number) and integer in order of their size.

- Task T24: Estimate the product of decimal numbers using reasoning without written multiplication and justify the estimate.

The comparison of the performance of procedural task T14 (M = 0.66, SD = 0.33, "*Computationally check whether the inequality* 9.5 · 99.8 > 1000 *holds true.*") and the performance of conceptual task T24 (M = 0.25, SD = 0.29, "*Without a written calculation, estimate whether the inequality 9.345 · 99.28 > 1000 holds true. Justify your answer.*") shows a statistically significant (z = -6.959, p < .001) higher performance in task T14 compared to task T24. In the written multiplication of two decimal numbers, two-thirds of the students were successful. The predominant difficulties observed in this task (T14) were in the placement of the decimal point (Figures 10 and 11) or inadequate multiplication execution. In individual cases, the student simply selects the calculation operation and writes the calculation but does not perform it (Figure 12).

In estimating and justifying the estimate of the product of two decimal numbers through reasoning, only a quarter of the students were successful. In justifying the estimate of the product (task T24), students were unable to reason that by comparing the whole part, they can estimate the product (e.g., the first factor is less than 10 and the second factor is less than 100, so the product is less than 1000). The student (Figure 14) did consider the whole parts of the decimal numbers in justifying the estimate of the product but omitted the decimal part and did not explain why: *"9.345 · 99.28 < 1000 because 9 · 99 = 891, and it is approximately 10 less than 1000."* Another difficulty in students' justification of the product estimation was that the justification was only a verbal definition of the task and not a reflection of conceptual understanding (Figure 13).

Table 5. Examples of Solved Tasks T1 and T2

Task T14 of Test T1

Difficulty computing the value of a numerical expression with decimal numbers
Without decimal points.

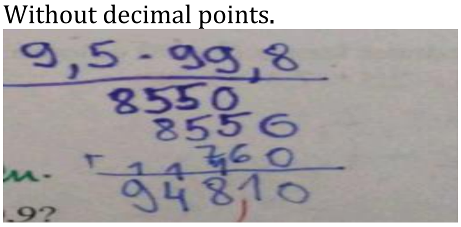


Figure 10. Example 1 of a Student's Difficulties with Procedural Knowledge

Difficulty computing the value of a numerical expression with decimal numbers
Incorrect decimal point placement.

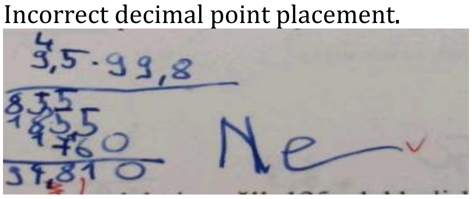


Figure 11. Example 2 of a Student's Difficulties with Procedural Knowledge

Difficulty computing the value of a numerical expression with decimal numbers
Selecting and writing down the correct calculation operation without performing the calculation.

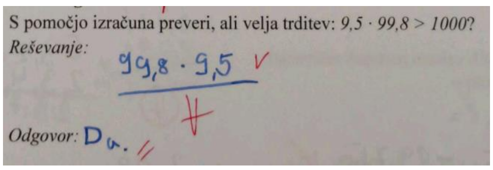


Figure 12. Example 3 of a Student's Difficulties with Procedural Knowledge

Task T24 of Test T2

Difficulty in the estimation of the product of two decimal numbers
The justification is only a verbalisation of the task, not a reflection of conceptual understanding.

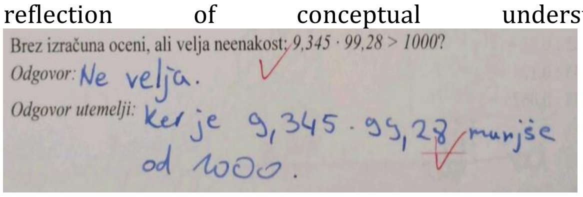


Figure 13. Example 1 of a Student's Difficulties with Conceptual Knowledge

Difficulty justifying the estimation of the product of two decimal numbers
When evaluating the product, only the integral part of the decimal numbers is taken into account, and the decimal part is omitted without explanation.

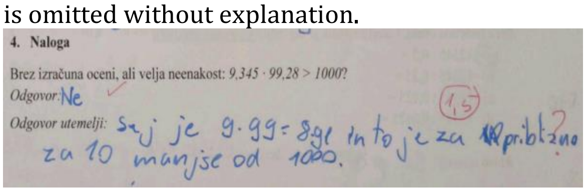


Figure 14. Example 2 of a Student's Difficulties with Conceptual Knowledge

d) Multiplying an integer with a decimal number between 0 and 1, with a decimal number greater than 1 and compare the two products

Examples of tasks:

- Task T15: Determine the product of an integer and a decimal number using written multiplication.
- Task T25: Estimate the product of an integer and a decimal number through reasoning, without written multiplication, and justify the estimate.

A comparison of the performance of the procedural task T15 (M = 0.70, SD = 0.38, *"Determine the greater product through calculation: 125 · 1.11 or 125 · 0.99?"*) with the performance of the conceptual task T25 (M = 0.25, SD = 0.37, *"Without a written calculation, estimate whether the inequality* 125 · 1.1 > 125 · 0.99 *holds true."*) shows a statistically significant (z = -4.885, p < .001) higher performance rate for procedural knowledge (T15) than for conceptual knowledge (T25). Even in the case of written multiplication of an integer by a decimal number (T15), the outstanding difficulty is the placement of the decimal point (Figure 15).

In the estimation of the product value and the justification of the estimate, we observed difficulties in explaining the estimate. Frequently, we only observed verbalization of the symbolic notation/algebraic expression (e.g., "Because 125 · 1.11 is greater than 125 · 0.99.") (Figure 16). Students who justified the problem correctly first observed that we multiply the same integer by two different decimals, where the first is less than one (0.99) and the second is greater than one (1.1). They then reasoned that multiplying the same integer by a larger decimal, results in a larger product.

Table 6. Examples of Solved Tasks T1 and T2

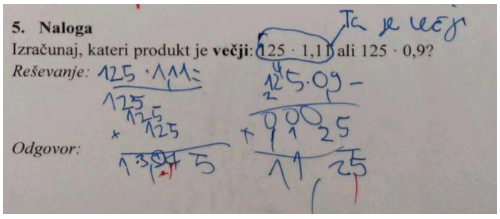
<table>
<tr><th>Task T15 of Test T1</th><th>Task T25 of Test T2</th></tr>
<tr><td>Difficulty computing the value of a numerical expression with decimal numbers
Incorrect decimal point placement.


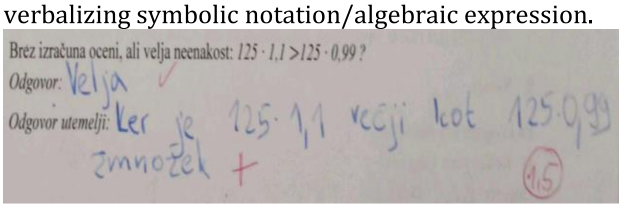
Figure 15. Example 1 of a Student's Difficulties with Procedural Knowledge</td><td>Difficulty justifying the estimation of the product of two decimal numbers
Difficulty in explanation: Justification is limited to verbalizing symbolic notation/algebraic expression.


Figure 16. Example 1 of a Student's Difficulties with Conceptual Knowledge</td></tr>
</table>

e) Dividing an integer with a decimal number between 0 and 1

Examples of tasks:

- Task T16: Determine the quotient between an integer and a decimal number using written division and compare quotients (decimal numbers) in order of their size.
- Task T26: Estimate the quotient of an integer and a decimal number through reasoning, without written division, and justify the estimate.

A comparison of the performance of the procedural task T16 (M = 0.40, SD = 0.42, *"Determine the smaller quotient through calculation: 125 : 0.2 or 125 : 0.25"*) with the performance of the conceptual task T26 (M = 0.25, SD = 0.28, *"Without a written calculation, estimate whether the inequality holds: 125 : 0.2 < 125 : 0.25 and justify the estimate"*), shows a statistically significant (z = -2.805, p = 0.005) higher performance rate for procedural knowledge (T16) than for conceptual knowledge (T26). The performance of tasks involving decimal division is lower compared to tasks involving the addition and multiplication of decimals. The difficulties encountered in tasks involving decimal division are similar to those in addition and multiplication of decimals but more frequent. Even when dividing decimals, students have difficulty writing division calculations and positioning decimal points (Figure 17).

In justifying the quotient of two decimal numbers (task T26), it is observed that the student (Figure 18) thinks about the size of the divisor but does not explain that dividing by a smaller divisor results in a larger quotient. Students often resort to recalculation or do not carry out justification in explaining their decisions.

Table 7. Examples of Solved Tasks T1 and T2

<table>
<tr><th>Task T16 of Test T1</th><th>Task T26 of Test T2</th></tr>
<tr><td>

Difficulty transforming text notation into numeric expression

Incorrect notation of calculation (omits the decimal point in the divisor, but does not increase the dividend tenfold).

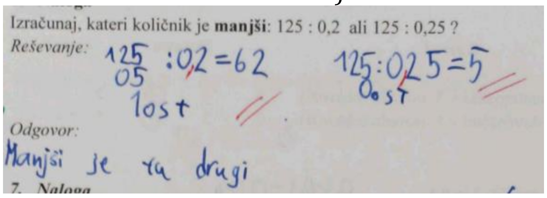


Figure 17. Example 1 of a Student's Difficulties with Procedural Knowledge

</td><td>

Difficulty justifying the estimation of the quotient of two decimal numbers

The student reflects on the size of the divisor but does not explain that as the divisor decreases, the quotient increases.

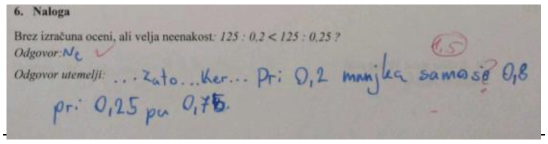


Figure 18. Example 1 of a Student's Difficulties with Conceptual Knowledge

</td></tr>
</table>

The noticeable conceptual difficulties in division are the misunderstanding of the relationship between the dividend and the divisor and the (lack of) understanding of the place values of decimal numbers. This is observed in both the conceptual task T26 (M = 0.26, SD = 0.28, *"Without a written calculation, estimate whether the inequality holds: 125 : 0.2 < 125 : 0.25 and justify the estimate."*), as well as in the conceptual task T28 (M = 0.30, SD = 0.34, *"Without a written calculation, estimate which of the following quotients 12345 : 0.5; 12345 : 0.25; 12345 : 0.125; 12345 : 0.0625 is the largest. Circle the correct one."*). Similar difficulties are observed (the student thinks about the size of the divisor but does not explain or adequately justify that when dividing by a smaller divisor, we get a larger quotient). Analysis of task T28 also shows that many students still have problems understanding place values, especially decimal values and numerical magnitude understanding and representations, as they believe that the decimal number 0.5 is the smallest among the numbers 0.5, 0.25, 0.125 and 0.0625. In this task, the latter represents more difficulty than understanding that dividing by a smaller divisor results in a larger quotient.

In the part that follows, we present an analysis of the selected task, presenting in particular the strategies used by the students. A Wilcoxon signed-rank test was not conducted for these tasks due to their lack of comparable goals in terms of content.

In task T17 (M = 0.67, SD = 0.28, "*Write a decimal number for a coloured area.*"), students had difficulties in understanding and distinguishing the concepts of decimal numbers and decimal fractions, where some students, instead of the required decimal number, wrote the fraction for a coloured area (Figure 19). In determining the part of the whole, they used different strategies: counting the coloured and all squares, drawing equal or proportional parts and giving only estimated values (e.g., one-third). In this process, individual students made mistakes in counting the coloured squares, and often, they counted the squares correctly but were unable to write the part of the whole (e.g., 68.00; 68; 6.8; 60.8). Two examples of writing part of the whole stand out, where the student wrote the coloured part as a whole part of the decimal number and the uncoloured part as a decimal part (68.32) or where the student wrote the coloured part as a whole part of the decimal number and the number of all squares as a decimal part (68.100).

Task T17

Write a decimal number for a coloured area

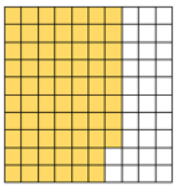


Figure 19. Understanding and Distinguishing the Concepts of Decimal Numbers and Decimal Fractions

The graphical representation of the part of the whole (Figure 20) also highlights problems in understanding the part of the whole and writing it as a fraction. For example, in the representation, they only consider the value of the numerator and ignore the denominator (instead of 1/10 they colour 1/100), and some students equate 1/10 with 1/2.

Colour 1/10 of the shape

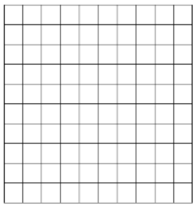


Figure 20. The Graphical Representation of the Part of the Whole

In addition, in other tasks that required justification, students also had difficulties, as the justification was either deficient, inadequate or based on incorrect or incomplete conceptual notions or students simply omitted justification. For example, in the task where the parts of a whole had to be read from a picture and compared in size and the size relationship between them had to be justified (Task T27, M = 0.75, SD = 0.26, Figure 21), students often compared the number of parts painted but did not state in the justification that the whole was the same in all three cases.

Task T27: *"Which shape has the largest coloured part? Circle the appropriate choice and justify it."*

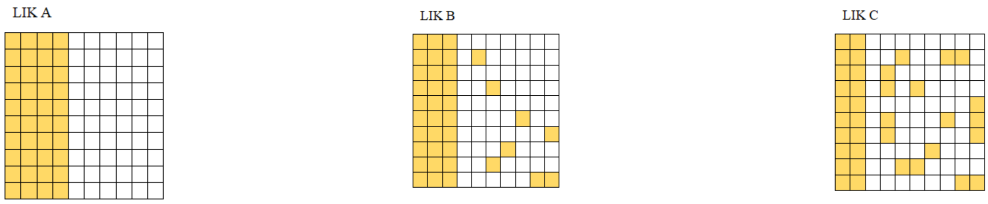


Figure 21. Choosing the Largest Coloured Part of the Whole

Discussion

This study confirms the findings of the literature review in different aspects. The results showed significant differences in students' performance in decimal numbers based on their conceptual knowledge and procedural knowledge. Despite the limitations of our current data collection methodology, which is based only on knowledge assessment tests, and may provide limited insight and understanding of participants' conceptual knowledge, similar to Kallai and Tzelgov (2014) students show a good knowledge of the procedural aspects of decimals, while their performance in conceptual understanding is much lower.

Difficulties identified in the research are diverse and relate to both procedural and conceptual knowledge, but significantly more students have difficulties at the level of conceptual knowledge. At the level of procedural knowledge, similar to various studies, including the work of Yang and Sianturi (2019a, 2019b), and McNeil and Alibali (2005), i.e. when performing computational operations between decimal numbers, we mainly observe difficulties in:

- transforming text notation into numerical expressions,
- incorrect choice of computational operation or incomplete or inappropriate notations,
- incorrect signing of decimal numbers in subtraction notations,
- omission of the decimal point in division notations or multiplying by the power of 10 only for the divisor and not equivalently for the divisor) and
- when calculating a numerical expression with decimal numbers (e.g., incorrect placement of the decimal point or omission of the decimal point when it is necessary).

At the level of conceptual knowledge, the results show that students have difficulties in:

- assessing and justifying the assessment of computational work without a written algorithm (student considers only the whole parts of the decimal number, leaving out the decimal part without explanation; the product of a decimal number greater than 1 and a decimal number between 0 and 1 is greater than the greater factor; decimal numbers between 0 and 1 are negative numbers, justification is limited to verbalizing the symbolic notation/algebraic expression; the student reflects on the size of the divisor, but does not explain that as the divisor decreases, the quotient increases), which is similar to findings of Deslis and Desli (2023),
- understanding and distinguishing between the concepts of decimal numbers and decimal fractions, similar to the findings of Girit and Akyuz (2016), and
- with mathematical vocabulary.

Advanced number sense, creativity, and flexibility are necessary for the effective use of computational estimation, mental calculation, or properties to assess reasonableness (Deslis & Desli, 2023; McMullen et al., 2020; NCTM, 2000).

At the level of conceptual knowledge, the results indicate the difficulty students have in estimating the sum, product and quotient of two decimal numbers with reflection, without written addition, multiplication and division, and in justifying these estimates. When justifying the magnitude of the sum or product, we observe that they often consider only the whole part of the decimal number and neglect the tens and hundreds, and do not explain why. When justifying, they often fall back on a written algorithm (auxiliary calculations), instead of assessing the value of the calculations by reflection or explaining them mathematically, with existing mathematical knowledge, in mathematical language. Students had difficulties understanding and distinguishing the concepts of decimal numbers and decimal fractions. The results point to a limited mathematical vocabulary.

At the level of conceptual knowledge, the results indicate that some of the students who were unsuccessful in multiplying a decimal greater than 1 by a decimal between 0 and 1, it is observed that some of them consider decimals between 0 and 1 to be negative numbers and that the product 3.28 * 0.12 is greater than the larger factor (e.g., 3.28 * 0.12 > 3.28). The study also showed that the students' estimations were more accurate when they included an integer and one decimal number (e.g., 125 * 1.11).

Our study is consistent with the findings of Sianturi et al. (2023), where a significant portion of students demonstrated proficiency in basic numerical operations but struggled with the application and reasoning of these concepts. This is further supported by the study of Lortie-Forgues and Siegler (2017), which found that students often face challenges in predicting the outcomes of operations involving decimal numbers, particularly when both decimals are less than 1.

In the study of conceptual knowledge, even more difficulties were observed among students when dividing decimal numbers. The majority of students have not yet mastered the relationship between the dividend and divisor and the place values, especially the decimal values and magnitude of numerical representations, as they think that a decimal number with more decimal places is larger. Similar findings were also discovered by Irwin (2001), as previously mentioned.

The results also indicate that students do not pay enough attention to the <, > symbols between numbers, which reflects in incorrect estimation. Difficulties are also observed in the notation of decimal numbers. Individual students write part of the whole with a decimal fraction instead of the required decimal.

The causes of students' learning difficulties with decimals are multifaceted and diverse. Even though "the Slovenian curriculum is designed to help students build the solid conceptual foundation in mathematics that will enable them to apply their knowledge and further their learning successfully" (Žakelj et al., 2011), the results of the survey suggest that teaching in Slovenian schools is still very much focused on solving mathematical problems, or on familiar mathematical tasks, on practising procedures, and less or not enough on exploration, reflection, assessment before computation, and reasoning, even though they may be an ideal opportunity to change misconceptions. Research consistently supports the value of concept-based learning in fostering both understanding and procedural knowledge (Chappell & Killpatrick, 2007; Higgins & Reid, 2017; Yuliandari & Anggraini, 2021).

We can assume that they may be due to learning and teaching approaches, students' insufficient experiences in the estimation and justification of tasks, or in curriculum or mathematics lesson planning etc. Based on the type of difficulties where we observe inadequate performance of computational algorithms, the results indicate that the reasons may be due to non-automatized addition and subtraction algorithms. More prominent are difficulties in conceptual knowledge, e.g., in estimation and justification. We can assume that difficulties may also arise from approaches to learning and teaching that prioritize learning procedures over the development of understanding of the size relationships between numbers that could be developed if students had more experience in estimation and justification.

The important role of language in learning and teaching mathematics cannot be ignored. Although the Slovenian curriculum emphasises the objectives of developing the ability to understand and analyse texts, to formulate questions from text, to express oneself clearly, to use clear mathematical language and to understand what is read, the results point to low achievement in reasoning or in formulating arguments in mathematical language, using mathematical terminology and symbolism, which may indicate that insufficient attention is being paid to achieving these objectives in teaching.

Given that the emphases that are crucial for the development of understanding of decimals are very explicit in the mathematics curriculum (Žakelj et al., 2011), it is assumed that the reason for the identified difficulties may be the relatively late introduction of decimals, in grade 6 of primary school, at the age of 11 to 12 years. By delaying the learning of these topics in the lower grades, students do not acquire the skills they need for their current daily lives, on the other hand, we miss the opportunity to gradually introduce decimals. The gradual introduction of concepts contributes to the development of conceptual knowledge, as conceptual schemes are formed gradually, supported by concrete experiences, from the concrete level, concrete real-life situations that support the students, to the stage where they start to learn about decimals at a more abstract level (e.g., understand decimal place values, recognize and write decimals using words, numbers and models; compare, order and round estimate the value of a numerical expression with reflection).

According to the TIMSS 2015 (Mullis et al., 2016) international study, many countries with high achievement introduce decimal numbers gradually and at a very early age. Students are first introduced to familiar decimal notations from everyday life, such as 0.5 litres or 1.5 kilogrammes, and to quantities of money written in decimals. In several countries, fourth-graders can then calculate with decimals, at least up to two decimal places. Special emphasis is placed on introducing children to 'adult', non-adapted money calculation, as children aged 10-11 need it urgently for their daily lives (Japelj Pavešić, 2017).

Conclusion

Students' difficulties in understanding decimals and calculating with decimals are complex and diverse. Procedural difficulties with decimals often stem from a lack of proficiency in basic arithmetic skills, such as addition, subtraction, multiplication, and division, which are necessary for effective manipulation of decimal numbers. These difficulties may also be related to a lack of conceptual knowledge, including a lack of familiarity with the base-10 number system, difficulty with place value, and difficulty with estimating values and justifying results. Using students' prior knowledge and real-world contexts is key to fostering the development of conceptual understanding in the area of decimals. These difficulties may result from approaches to learning and teaching that prioritize the acquisition of procedures over the development of an understanding of the size relationships between numbers, which could be developed through more experiences with an estimation of value/result and justification.

Quantitative and qualitative analysis of the research results shows that the students included in the research perform better in selecting and performing calculation procedures for adding, multiplying, and dividing decimal numbers, in selecting and using strategies for solving procedural tasks with decimal numbers than conceptual tasks, in expressing understanding of size relationships among numbers, estimating (values) and providing justification. Estimating the sum, product, and quotient with reflection, justifying these estimations with mathematical facts, and justifying these estimations in mathematical language stands out as particularly difficult.

The role of language in the learning and teaching of mathematics should also be taken into account, as students' performance in reasoning and expressing answers in mathematical language using terminology and symbolism is often poor. This may be an indication that not enough attention is being paid to these objectives in the classroom.

To overcome such difficulties, more balance between procedural and conceptual knowledge in the process of teaching and learning mathematics is needed. Hiebert and Lefevre (1986) note that procedural and conceptual knowledge are fundamental components of mathematical knowledge, but it is only the ability to integrate both types of knowledge that enables an individual to develop a deeper understanding of concepts and procedures. This means that students can carry out mathematical procedures as well as explain and justify the meaning of the solutions they have reached through these procedures. The NCTM (2000) document highlights the positive effects of balancing procedural and conceptual knowledge in teaching and learning mathematics, including the development of students' numerical representations as well as their problem-solving skills.

As mentioned above, the results of the survey highlighted difficulties in students' procedural knowledge of decimals, but the difficulties in developing conceptual knowledge were much more pronounced. Difficulties in estimating and justifying the sum, product and quotient of decimals stand out, which we perceived as the students' outstanding difficulties in decimals.

A key aspect contributing to these difficulties is the utilisation of inadequate teaching and learning methods (Japelj Pavešić, 2017) that do not provide students with a diverse set of activities, such as estimate and reasoning tasks, and instead, prioritise the development of procedural knowledge. As Yuliandari and Anggraini (2021) state, in developing conceptual understanding it is important to use clear and accessible language in teaching. This requires using questions that encourage students to think, avoiding learning shortcuts, avoiding memorizing facts and procedures, helping students make connections between concepts, and developing understanding through a three-stage learning process (Concrete-Representation-Abstract).

Recommendations

Further research could be extended to investigate teachers' competencies for teaching mathematics at the level of the development of conceptual knowledge of decimals and also to explore the role and influence of procedural and conceptual knowledge on developing problem-solving competencies.

To facilitate effective teaching practice, the teacher should anticipate the learner's knowledge pathways from the prior knowledge to the new. In the context of teaching decimals, the teacher needs to know how decimals are related to the decimal system, whole numbers and equivalent fractions. The intersection points of these mathematical constructions are many and the connections can be made in the classroom in many different ways. The teacher's role extends beyond passive reception, it involves active participation in students' discourse: listening carefully to students' contributions and assessing whether their statements are accurate, based on knowledge, based on accepted definitions, and leading to the desired conclusion and inference. For effective learning and teaching of decimals, it is useful to provide students with a variety of learning and technical aids that can serve, among other things, as a cognitive tool, as a support for illustrating concepts and relationships, as an aid to understanding, as a support in the learning process, as a reminder of the steps of solving, as a sense of security or as a motivational tool.

Limitations

The present study has several constraints that can be used as a starting point for future research. The study was conducted in six existing randomly selected 6th grade classes from six Slovenian schools, including a sample size of one hundred children, which represents approximately 0.5 % of all Slovenian 6th graders. This means that we have not previously tested the effects of teaching and learning approaches on the development of conceptual and procedural knowledge of decimals. We also did not test the use of teaching materials.

A limitation of our current data collection methodology is also that it is based on knowledge assessment tests only. While these tests provide valuable insights into participants' performance, they have limitations in capturing nuanced aspects of conceptual understanding. To overcome this limitation and increase the comprehensiveness of our research, we propose the inclusion of unstructured interviews as a possible way to gain clarification and more in-depth insight into participants' conceptual knowledge. This dual approach, combining knowledge assessment tests with unstructured interviews, could provide a more comprehensive and richer view of participants' conceptual understanding.

As the study was conducted in Slovenia, in line with the curriculum standards of the Slovenian math curriculum as mandated by the Ministry of Education, future studies should verify these findings in other school systems.

All of these constraints could be used as a starting point for future research.

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Authorship Contribution Statement

Žakelj: Conceptualization, design, data acquisition, data analysis / interpretation, drafting manuscript, critical revision of manuscript, supervision. Klančar: Conceptualization, design, data acquisition, data analysis / interpretation, drafting manuscript, critical revision of manuscript, statistical analysis.

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